# 1 Balance of angular momentum in a non-inertial frame

Traditionally, the LE is derived by transforming the conservation of angular momentum into the rotating (i.e. non-inertial) body-fixed reference frame:

$$\boldsymbol{M} = \frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{H} = \frac{\mathrm{d}(\boldsymbol{I} \cdot \boldsymbol{\omega} + \boldsymbol{h})}{\mathrm{d}t} + \boldsymbol{\omega} \times (\boldsymbol{I} \cdot \boldsymbol{\omega} + \boldsymbol{h}) \tag{1}$$

where $\boldsymbol{M}$ is the moment of external forces, and the Resal's theorem was used to express the time derivative of the total angular momentum $\boldsymbol{H}$. The Resal's theorem holds for an arbitrary vector $\boldsymbol{a}$: as long as the vector is the same for each time $t$ in both frames, its rate differs according to $\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}|_{\mathrm{non-rot}} = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}|_{\mathrm{rot}} + \boldsymbol{\omega} \times \boldsymbol{a}$. The relative angular momentum in the body-fixed frame was denoted as $\boldsymbol{h}$:

$$\boldsymbol{h} := \int_{v(t)} \boldsymbol{r} \times \boldsymbol{v}\, \rho \, \mathrm{d}v, \tag{2}$$

with $\boldsymbol{r}$ is the position vector. Below we test whether the balance of angular momentum holds also in the body-fixed frame, i.e. whether the rate of angular momentum, as measured in the body-fixed frame, is equal to the torque of all the forces felt in the body-fixed frame:

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} \stackrel{?}{=} \int_{v(t)} \boldsymbol{r} \times \boldsymbol{f}\, \rho \, \mathrm{d}v, \tag{3}$$

where $v(t)$ is the volume occupied by the body, and a stress-free surface is assumed. In a non-inertial reference frame, the so-called fictitious forces have to be included as additional body forces $\boldsymbol{f}$. Therefore, $\boldsymbol{f}$ can be separated into external (e.g. the gravitational force due to an external body), internal (e.g. the gravitational force due to mass of the body itself), and fictitious forces: $\boldsymbol{f} = \boldsymbol{f}_{\mathrm{ext}} + \boldsymbol{f}_{\mathrm{int}} + \boldsymbol{f}_{\mathrm{fic}}$. After identifying $\int_{v(t)} \boldsymbol{r} \times \boldsymbol{f}_{\mathrm{ext}} \rho \, \mathrm{d}v$ as the moment of external forces $\boldsymbol{M}$, we get:

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \boldsymbol{M} + \int_{v(t)} \boldsymbol{r} \times \boldsymbol{f}_{\mathrm{int}}\, \rho \, \mathrm{d}v + \int_{v(t)} \boldsymbol{r} \times \boldsymbol{f}_{\mathrm{fic}}\, \rho \, \mathrm{d}v \tag{4}$$

When all the internal forces are central, the second term on the RHS of Eq. (4) is equal to zero. The fictitious forces are (Martinec, 2019):

$$\boldsymbol{f}_{\mathrm{fic}} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - 2\boldsymbol{\omega} \times \boldsymbol{v} - \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} - \frac{d\boldsymbol{v}_0}{dt} - \boldsymbol{\omega} \times \boldsymbol{v}_0 \tag{5}$$

where $\boldsymbol{v}_0$ is the velocity of the origin of the body-fixed frame relative to the inertial reference frame. The last two forces are uniform, and thus their integral contributions to Eq. (4) are equal to zero, because in our case the origin of the non-inertial frame lies in the center of mass. Besides $\boldsymbol{M}$, the only remaining contributions on the RHS of Eq. (4) come from the centrifugal, Coriolis, and Euler forces. Their relative contributions can be written as follows.

Centrifugal force:

$$-\int_{v(t)} \boldsymbol{r} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}))\, \rho \, \mathrm{d}v = -\int_{v(t)} \boldsymbol{r} \times (\boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\omega} \cdot \boldsymbol{\omega}))\, \rho \, \mathrm{d}v =$$
$$-\int_{v(t)} \boldsymbol{r} \times \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{r})\, \rho \, \mathrm{d}v = \boldsymbol{\omega} \times \int_{v(t)} \boldsymbol{r}\, (\boldsymbol{\omega} \cdot \boldsymbol{r})\, \rho \, \mathrm{d}v = -\boldsymbol{\omega} \times (\boldsymbol{I} \cdot \boldsymbol{\omega})$$

Coriolis force:

$$-\int_{v(t)} \boldsymbol{r}\times(2\boldsymbol{\omega}\times\boldsymbol{v})\,\rho\,\mathrm{d}v = \int_{v(t)} (\boldsymbol{v}(\boldsymbol{r}\cdot 2\boldsymbol{\omega}) - 2\boldsymbol{\omega}(\boldsymbol{r}\cdot\boldsymbol{v}))\,\rho\,\mathrm{d}v =$$

$$-\int_{v(t)} (\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{v}) - \boldsymbol{v}(\boldsymbol{r}\cdot\boldsymbol{\omega}))\,\rho\,\mathrm{d}v - \int_{v(t)} 2\boldsymbol{\omega}(\boldsymbol{r}\cdot\boldsymbol{v}) - \boldsymbol{r}(\boldsymbol{v}\cdot\boldsymbol{\omega}) - (\boldsymbol{v}(\boldsymbol{r}\cdot\boldsymbol{\omega}))\,\rho\,\mathrm{d}v =$$

$$-\int_{v(t)} \boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{v})\,\rho\,\mathrm{d}v - \left(\int_{v(t)} 2(\boldsymbol{r}\cdot\boldsymbol{v})\,\mathbb{1} - \boldsymbol{r}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{r})\,\rho\,\mathrm{d}v\right)\cdot\boldsymbol{\omega} =$$

$$-\boldsymbol{\omega}\times\int_{v(t)} \boldsymbol{r}\times\boldsymbol{v}\,\rho\,\mathrm{d}v - \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}\cdot\boldsymbol{\omega} = -\boldsymbol{\omega}\times\boldsymbol{h} - \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}\cdot\boldsymbol{\omega}$$

Euler force:

$$-\int_{v(t)} \boldsymbol{r}\times(\frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r})\,\rho\,\mathrm{d}v = -\int_{v(t)} \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}r^2 - \boldsymbol{r}(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\cdot\boldsymbol{r})\,\rho\,\mathrm{d}v = -\boldsymbol{I}\cdot\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}$$

Upon inserting these contributions into Eq. (4), the LE is finally obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{I}\cdot\boldsymbol{\omega}+\boldsymbol{h}) + \boldsymbol{\omega}\times(\boldsymbol{I}\cdot\boldsymbol{\omega}+\boldsymbol{h}) = \boldsymbol{M} \tag{6}$$

The one to last equalities in the above expressions for the centrifugal, Coriolis, and Euler forces come from the definition of the inertia tensor,

$$\boldsymbol{I} = \int_{v(t)} ((\boldsymbol{r}\cdot\boldsymbol{r})\,\mathbb{1} - \boldsymbol{r}\otimes\boldsymbol{r})\rho\,\mathrm{d}v, \tag{7}$$

where $\mathbb{1}$ is the unit tensor and $\otimes$ is the tensor product, and from the time derivative of $\boldsymbol{I}$ (Martinec, 2019),

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \int_{v(t)} [2(\boldsymbol{r}\cdot\boldsymbol{v})\,\mathbb{1} - \boldsymbol{r}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{r}]\,\rho\,\mathrm{d}v \tag{8}$$

where the latter can be derived by employing Reynold's transport theorem to Eq. (7).

From the above derivation of the LE it is immediately clear that it is the Coriolis and Euler forces that combine to yield the time derivative terms that appear in the LE. Consequently, assuming that the centrifugal force is the only fictitious force in the equation of motion directly yields the $\omega||$MIA approximation,

$$\boldsymbol{\omega}\times(\boldsymbol{I}\cdot\boldsymbol{\omega}) + \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \boldsymbol{M} \tag{9}$$

Therefore, the range of validity of the $\omega||$MIA approximation is related to the importance of the Euler and Coriolis forces in shaping the body's inertia tensor. Note that the LE is typically solved in the Tisserand frame, i.e. in the frame with zero relative angular momentum $\boldsymbol{h}$ (Munk and MacDonald, 1960).

## 2 Energy balance in a non-inertial frame

Rotational potential for a general orientation of the rotation vector $\boldsymbol{\omega}$ can be written as

$$\Psi = \frac{1}{2}\left((\boldsymbol{\omega}\cdot\boldsymbol{r})^2 - |\boldsymbol{\omega}|^2|\boldsymbol{r}|^2\right), \tag{10}$$

which gives, for the special case of $\boldsymbol{\omega}$ being coincident with the $z$-axis,

$$\Psi = -\frac{1}{2}\omega^2 r^2\sin^2\theta = \frac{1}{3}\omega^2 r^2 P_2^0(\cos\theta) - \frac{1}{3}\omega^2 r^2, \tag{11}$$

with colatitude $\theta$. The tidal potential can, for the case of tidal axis parallel to the $z{-}axis$, be written as

$$\Theta = -\frac{GM}{a^3}r^2 P_2^0(\cos\theta) = -\frac{1}{3}|\overline{\boldsymbol{\omega}}|^2 r^2 P_2^0(\cos\theta) \qquad \left(= \frac{1}{6}|\overline{\boldsymbol{\omega}}|^2 r^2 - \frac{1}{2}(\overline{\boldsymbol{\omega}}\cdot\boldsymbol{r})^2\right), \tag{12}$$

which means that it has the same form as the degree two component of the centrifugal potential, differing only in amplitude and sign. In the second equality we have, for convenience, introduced a tidal vector, $\overline{\boldsymbol{\omega}}$, with amplitude given as $\sqrt{3GM/a^3}$, where $M$ is mass of the central body.

Conservation of energy in a rotating frame can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v(t)}\left(\rho\epsilon+\tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\right)\,\mathrm{d}v=\int_{v(t)}\left(-\nabla U-\nabla\Theta-\nabla\Psi-2\boldsymbol{\omega}\times\boldsymbol{v}-\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{r}\right)\cdot\boldsymbol{v}\,\rho\,\mathrm{d}v. \tag{13}$$

The power of the Coriolis force is identically zero, and in the Tisserand frame also the integrated power of the Euler force is zero:

$$\int_{v(t)}\left(-\rho\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{r}\right)\cdot\boldsymbol{v}\,\mathrm{d}v=-\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\cdot\int_{v(t)}\boldsymbol{r}\times(\rho\boldsymbol{v})\,\mathrm{d}v=-\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\cdot\boldsymbol{h}\;=\boldsymbol{0}, \tag{14}$$

For the power of the centrifugal force one gets:

$$\int_{v(t)}-(\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}))\cdot\boldsymbol{v}\,\rho\,\mathrm{d}v=\int_{v(t)}\left[(\boldsymbol{r}\cdot\boldsymbol{v})\,\omega^2-(\boldsymbol{\omega}\cdot\boldsymbol{v})\cdot(\boldsymbol{\omega}\cdot\boldsymbol{r})\right]\,\rho\,\mathrm{d}v=\frac{1}{2}\boldsymbol{\omega}\cdot\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}\cdot\boldsymbol{\omega} \tag{15}$$

Analogously to Eq. (15), for the power of the tidal force one gets:

$$-\int_{v(t)}\nabla\Theta\cdot\boldsymbol{v}\,\rho\,\mathrm{d}v=\int_{v(t)}\left[(\overline{\boldsymbol{\omega}}\cdot\boldsymbol{v})\cdot(\overline{\boldsymbol{\omega}}\cdot\boldsymbol{r})-\frac{1}{3}(\boldsymbol{r}\cdot\boldsymbol{v})\,|\overline{\boldsymbol{\omega}}|^2\right]\,\rho\,\mathrm{d}v=-\frac{1}{2}\overline{\boldsymbol{\omega}}\cdot\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}\cdot\overline{\boldsymbol{\omega}}-\frac{\mathrm{d}}{\mathrm{d}t}\int_{v(t)}\frac{1}{3}r^2|\overline{\boldsymbol{\omega}}|^2\rho\;\mathrm{d}v$$

After employing Eq. (A21) from https://doi.org/10.1093/gji/ggx469, and after using the above derived expression for the power of the centrifugal force, Eq. (13) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v(t)}\left(\epsilon+\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}+\frac{1}{2}U\right)\,\rho\,\mathrm{d}v=\frac{1}{2}\boldsymbol{\omega}\cdot\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}\cdot\boldsymbol{\omega}-\int_{v(t)}\nabla\Theta\cdot\boldsymbol{v}\rho\,\mathrm{d}v. \tag{16}$$

If one wants the total energy of the system to be on LHS, the rotational energy must be added on both sides:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{v(t)}\left(\epsilon+\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}+\frac{1}{2}U-\Psi\right)\,\rho\,\mathrm{d}v=\frac{1}{2}\boldsymbol{\omega}\cdot\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t}\cdot\boldsymbol{\omega}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}\right)-\int_{v(t)}\nabla\Theta\cdot\boldsymbol{v}\rho\,\mathrm{d}v, \tag{17}$$

and so, after evaluating the last term and recalling the symmetry of $\boldsymbol{I}$,

$$E_{\mathrm{spec}}+E_{\mathrm{kin}}+E_{\mathrm{grav}}+E_{\mathrm{rot}}=\frac{\mathrm{d}(\boldsymbol{I}\cdot\boldsymbol{\omega})}{\mathrm{d}t}\cdot\boldsymbol{\omega}-\int_{v(t)}\nabla\Theta\cdot\boldsymbol{v}\rho\,\mathrm{d}v. \tag{18}$$

And the "tidal energy" we define as:

$$E_{\mathrm{tid}}:=\int_{v(t)}\Theta\rho\;\mathrm{d}v=\int_{v(t)}\left(\frac{1}{6}r^2|\overline{\boldsymbol{\omega}}|^2-\frac{1}{2}(\overline{\boldsymbol{\omega}}\cdot\boldsymbol{r})^2\right)\rho\;\mathrm{d}v=\frac{1}{2}\overline{\boldsymbol{\omega}}\cdot\boldsymbol{I}\cdot\overline{\boldsymbol{\omega}}-\int_{v(t)}\frac{1}{3}r^2|\overline{\boldsymbol{\omega}}|^2\,\rho\;\mathrm{d}v. \tag{19}$$

Putting all the system energies on the RHS on Eq. (22),

$$E_{\mathrm{spec}}+E_{\mathrm{kin}}+E_{\mathrm{grav}}+E_{\mathrm{rot}}+E_{\mathrm{tid}}=\frac{\mathrm{d}(\boldsymbol{I}\cdot\boldsymbol{\omega})}{\mathrm{d}t}\cdot\boldsymbol{\omega}+(\boldsymbol{I}\cdot\overline{\boldsymbol{\omega}})\cdot\frac{\mathrm{d}\overline{\boldsymbol{\omega}}}{\mathrm{d}t}, \tag{20}$$

which is a generally valid energy balance in a rotating frame. Now, if the angular moment conservation law is also applied, and projected onto $\boldsymbol{\omega}$,

$$\boldsymbol{M}\cdot\boldsymbol{\omega}=\left(\frac{\mathrm{d}(\boldsymbol{I}\cdot\boldsymbol{\omega})}{\mathrm{d}t}+\boldsymbol{\omega}\times(\boldsymbol{I}\cdot\boldsymbol{\omega})\right)\cdot\boldsymbol{\omega}=\frac{\mathrm{d}(\boldsymbol{I}\cdot\boldsymbol{\omega})}{\mathrm{d}t}\cdot\boldsymbol{\omega}, \tag{21}$$

we see that, for the case when the balance of angular momentum is enforced, one gets

$$E_{\mathrm{spec}}+E_{\mathrm{kin}}+E_{\mathrm{grav}}+E_{\mathrm{rot}}+E_{\mathrm{tid}}=\boldsymbol{M}\cdot\boldsymbol{\omega}+(\boldsymbol{I}\cdot\overline{\boldsymbol{\omega}})\cdot\frac{\mathrm{d}\overline{\boldsymbol{\omega}}}{\mathrm{d}t}=(\boldsymbol{I}\cdot\overline{\boldsymbol{\omega}})\cdot\left(\boldsymbol{\omega}\times\overline{\boldsymbol{\omega}}+\frac{\mathrm{d}\overline{\boldsymbol{\omega}}}{\mathrm{d}t}\right) \tag{22}$$

where $\boldsymbol{M}$ is the moment of external forces acting on the system, i.e. $\overline{\boldsymbol{\omega}}\times(\boldsymbol{I}\cdot\overline{\boldsymbol{\omega}})$ for the tidal forcing, and where the vector identity $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ was applied.

# 3 The linearized Liouville equations (LLE)

Munk and MacDonald (1960) proposed the following perturbation scheme to be plugged into the Liouville equation (notation is slightly modified from the one in the book):

$$\begin{array}{ccc} I_{11}^{\text{total}} = A + I_{11} & I_{22}^{\text{total}} = A + I_{22} & I_{33}^{\text{total}} = C + I_{33} \\ I_{12} = I_{12} & I_{13} = I_{13} & I_{23} = I_{23} \\ \omega_1 = \Omega m_1 & \omega_2 = \Omega m_2 & \omega_3 = \Omega(1 + m_3) \end{array}$$

If we neglect the products and squares of $\frac{I_{ij}}{C}, m_i, \frac{h_i}{\Omega C}$, as they are small dimensionless numbers (for the case of the Earth), the original Liouville equations decouple into two linear ones:

$$i\frac{\dot{\mathbf{m}}}{\sigma_r} + \mathbf{m} = \Phi, \quad \dot{m}_3 = \dot{\phi}_3, \quad \text{where} \quad \sigma_r := \frac{C - A}{A}\Omega \tag{23}$$

The first one is called the polar motion (PM) equation, the second length of day (LOD) equation, for that is what they govern. Complex notation was introduced: $\mathbf{m} = m_1 + im_2, \quad \Phi = \phi_1 + i\phi_2$, with the excitation functions $\phi_1, \phi_2, \phi_3$ having the following form:

$$\begin{aligned} \Omega^2(C - A)\phi_1 &= \Omega^2 I_{13} + \Omega\dot{I}_{23} + \Omega h_1 + \dot{h}_2 - L_2 \\ \Omega^2(C - A)\phi_2 &= \Omega^2 I_{23} + \Omega\dot{I}_{13} + \Omega h_2 + \dot{h}_1 - L_1 \\ \Omega^2 C\phi_3 &= -\Omega^2 I_{33} - \Omega h_3 - \Omega\int_0^t L_3 \,\mathrm{d}t \end{aligned}$$

The excitation function has to be computed for the studied process (e.g. loading, winds) in order to determine its effect on polar wander. The equation 23 can also be written in the form:

$$i\frac{\dot{\mathbf{m}}}{\sigma_r} + \mathbf{m} = \Psi - \frac{i}{\Omega}\dot{\Psi} + \frac{L}{(C - A)\Omega^2}, \tag{24}$$

where angular excitation function $\Psi = \psi_1 + i\psi_2$ was defined as

$$\psi_1 = \frac{1}{(C - A)\Omega}\left(\Omega I_{13} + h_1\right), \quad \psi_2 = \frac{1}{(C - A)\Omega}\left(\Omega I_{23} + h_2\right) \tag{25}$$

and complex notation $L = L_1 + iL_2$ was introduced.

In the scheme above, long-term rotational deformation (the "fluid" limit) is represented by the reference values $A$ and $C$ in 3 ($A, A, C$ being the principal moments of inertia for a zero perturbation). The instantaneous elastic response to centrifugal potential that causes an instantaneous (but only partial) readjustment of the equatorial bulge, can be expressed as (again neglecting the squares and products of $m_i$):

$$\begin{aligned} I_{11}^R = I_{22}^R = -\frac{2k_2^T}{3k_2^f}(C - A)m_3, \quad I_{33}^R = \frac{4k_2^T}{3k_2^f}(C - A)m_3 \\ I_{13}^R = \frac{k_2^T}{k_2^f}(C - A)m_1, \quad I_{23}^R = \frac{k_2^T}{k_2^f}(C - A)m_2 \end{aligned}$$

The free wobble is then obtained when excitation due to solely the elastic rotational deformation is considered (such excitation is unavoidable), with the PM equation (23) becoming:

$$i\dot{\mathbf{m}} + \sigma_r\mathbf{m} = \frac{k_2^T}{k_2^f}\sigma_r\boldsymbol{m} \quad \Rightarrow \quad i\dot{\mathbf{m}} + \sigma_{Ch}\mathbf{m} = 0, \tag{26}$$

where the Chandler frequency $\sigma_{Ch}$ has simple relation to the Euler frequency $\sigma_r$:

$$\sigma_{Ch} := \sigma_r\frac{k_f - k_2^T}{k_f} \tag{27}$$