

1 Thermal Convection (Rayleigh-Bénard)

The Eulerian equations of motion in the Boussinesq approximation are:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p / \rho_0 + \nu \nabla^2 \mathbf{u} - \alpha(T - T_0) \mathbf{g}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\partial_t T + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T. \quad (3)$$

The pre-stress $\nabla p_0 = \rho_0 \mathbf{g}$ is already subtracted in eq. (1), i.e. p is the dynamic pressure. The dimensional parameters of the system are: ν is kinematic viscosity, T_0 is the reference temperature, ρ_0 is the mean mass density at the reference temperature T_0 , α is the volumetric thermal expansion coefficient with respect to the reference temperature, responsible for the buoyancy term, $\rho(T) \simeq \rho_0(1 - \alpha(T - T_0))$, g is the amplitude of the gravitational acceleration field, and κ is the thermal diffusivity.

Introducing non-dimensional units for length, velocity, and temperature:

$$\mathbf{X} = \mathbf{x} / H \quad (4)$$

$$\mathbf{U} = \mathbf{u} / U_f \quad \text{with} \quad U_f = \sqrt{\alpha g \Delta T H} \quad (5)$$

$$\theta = (T - T_0) / \Delta T \quad (6)$$

Non-dimensionalization of time and pressure then follows from the choice above:

$$\tau = t / \tau_H \quad \text{and} \quad P = \frac{p}{\rho_0 U_f^2}, \quad \text{where} \quad \tau_H = H / U_f. \quad (7)$$

The procedure for the non-dimensionalization of the equations of motion is the following:

$$\frac{d(U U_f)}{d(H\tau/U_f)} + (U U_f) \frac{d(U U_f)}{d(XH)} = -\frac{dp}{d(XH)} \frac{1}{\rho_0} + \nu \frac{d^2(U U_f)}{d(XH)^2} - \alpha g (T - T_0), \quad / \cdot \frac{H}{U_f^2} \quad (8)$$

$$\partial_\tau \mathbf{U} + (\mathbf{U} \cdot \nabla_X) \mathbf{U} = -\nabla_X \left(\frac{p}{\rho_0 U_f^2} \right) + \frac{\nu}{H U_f} \nabla_X^2 \mathbf{U} + \theta \hat{\mathbf{z}}, \quad (9)$$

which is eq. (12). Similarly, the non-dimensional heat equation was derived as

$$\frac{d(\theta \Delta T + T_0)}{d(H\tau/U_f)} + (U U_f) \frac{d(\theta \Delta T + T_0)}{d(XH)} = \kappa \frac{d^2(\theta \Delta T + T_0)}{d(XH)^2}, \quad / \cdot \frac{H}{\Delta T U_f} \quad (10)$$

$$\partial_\tau \theta + (\mathbf{U} \cdot \nabla_X) \theta = \frac{\kappa}{H U_f} \nabla_X^2 \theta, \quad (11)$$

which is eq. (14), because $\sqrt{Pr Ra} = H U_f / \kappa$ (and the derivatives of the constant T_0 were evaluated as zero).

In terms of the non-dimensional quantities the governing equations read:

$$\partial_\tau \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{U} + \theta \hat{\mathbf{z}} \quad (12)$$

$$\nabla \cdot \mathbf{U} = 0 \quad (13)$$

$$\partial_\tau \theta + (\mathbf{U} \cdot \nabla) \theta = \frac{1}{\sqrt{Pr Ra}} \nabla^2 \theta \quad (14)$$

where $Ra = \alpha g \Delta T H^3 / (\nu \kappa)$ and $Pr = \nu / \kappa$. Therefore, the dimensionless control parameters of the system are Ra , Pr and Γ . Note that

$$\sqrt{\frac{Pr}{Ra}} = \sqrt{\frac{\nu \nu \kappa}{\kappa \alpha g \Delta T H^3}} = \frac{\nu}{\sqrt{\alpha g \Delta T H}} = \frac{\nu}{U_f H} = \frac{1}{Re} \quad (15)$$

gives the relation between the Rayleigh number Ra , Prandtl number Pr , and the Reynolds number Re .

Note that the non-dimensionalization above differs from the **geodynamic non-dimensionalization**, where the characteristic velocity is typically chosen as $U_d = \kappa/H$ (the time is scaled according to the diffusive time-scale H^2/κ , the length by H , and the velocity scaling then follows from these). In this case we can no longer use the relation $U_f^2 = \alpha \Delta T g H$ and the procedure for the equation of motion becomes:

$$\frac{d(U U_d)}{d(H\tau/U_d)} + (U U_d) \frac{d(U U_d)}{d(XH)} = -\frac{dp}{d(XH)} \frac{1}{\rho_0} + \nu \frac{d^2(U U_d)}{d(XH)^2} - \alpha g(T - T_0), \quad / \cdot \frac{H^3}{\kappa \nu} \quad (16)$$

$$\frac{\kappa}{\nu} (\partial_\tau \mathbf{U} + \mathbf{U} \cdot \nabla_X \mathbf{U}) = -\nabla_X \left(\frac{p H^2}{\rho_0 \kappa \nu} \right) + \nabla_X^2 \mathbf{U} - \frac{\alpha \Delta T g H^3}{\kappa \nu} \theta, \quad (17)$$

$$\frac{1}{Pr} \left(\frac{D\mathbf{U}}{D\tau} + \nabla_X P \right) = \nabla_X^2 \mathbf{U} + Ra \theta \hat{\mathbf{z}}. \quad (18)$$

The gradient of the non-dimensional pressure $P = p H^2 / (\rho_0 \kappa^2) = p / (\rho_0 U_d^2)$ was moved to the LHS. In the $Pr \rightarrow \infty$ limit, the pressure gradient disappears from the governing equation.

Non-dimensionalization vs scaling. Eq. (18) differs from what you find in geodynamical textbooks: Textbooks assume the non-dimensional pressure $P_g := p/C_{pg}$ (the letter g for “geodynamical”), with the conversion factor C_{pg} defined as

$$C_{pg} := \nu \frac{\rho_0 U_d}{H} \quad (19)$$

where ν^* is the non-dimensional kinematic viscosity. It differs from the mere unit conversion procedure: $C_p = C_\rho C_u^2 = \rho_0 U_d^2$, that results from observing the units of quantities: $\text{Pa} = \text{kg}/(\text{ms}^2) = (\text{kg}/\text{m}^3)(\text{m}^2/\text{s}^2)$. In terms of P_g instead of P , the governing equations read:

$$\frac{1}{Pr} \frac{D\mathbf{U}}{D\tau} = -\nabla_X P_g + \nabla_X^2 \mathbf{U} + Ra \theta \hat{\mathbf{z}}. \quad (20)$$

Note that in the $Pr \rightarrow \infty$ limit the non-dimensional pressure P_g goes to zero: $P_g = P/\nu^* = P\kappa/Pr \rightarrow 0$ for $Pr \rightarrow \infty$ if the limit is achieved by increasing the kinematic viscosity of the fluid while keeping the heat diffusivity constant – this explains the apparent mystery of how it is possible that in one formulation the pressure gradient is and in the other it disappears in the $Pr \rightarrow \infty$ limit.

It is because we are in fact making an assumption about scaling:

$$C_{pg} := \nu \frac{\rho_0 U_d}{H} = \nu^* H U_d \frac{\rho_0 U_d}{H} = \nu^* \rho_0 U_d^2, \quad (21)$$

where ν^* is the non-dimensional viscosity, as obtained from a mere unit conversion. This is why the textbooks justify the use of P_g by saying “by assuming comparability of pressure and viscous forces, the scale for the dynamic pressure is...” – we are in fact making an assumption that if the viscosity is higher, the dynamic pressure in the system will also be higher. We are thus making an assumption about how the pressure gradient **scales** in the system.

