

Supplementary material for “Fast and cheap approximation of Green functions uncertainty for waveform-based earthquake source inversions” by Miroslav Hallo and František Gallovič

SUPPLEMENT A: Derivation of stationarized approximate covariance functions (SACF and SAXCF)

In the main text of the article, we define the stationarized auto-covariance $\overline{cov}$ as

$$\overline{cov}(\tau) \stackrel{def}{=} \frac{1}{T} \int_{-\infty}^{\infty} \widetilde{cov}(t, \tau) dt, \quad (\text{A.1})$$

where the constant T is the duration of the dominant part of the signal (wavelet duration). The stationarization removes time dependence of the covariance and consider only its mean over time as a representative value. Similarly, we define the stationarized cross-covariance $\overline{xcov}$:

$$\overline{xcov}(\tau) \stackrel{def}{=} \frac{1}{T} \int_{-\infty}^{\infty} \widetilde{xcov}(t, \tau) dt. \quad (\text{A.2})$$

There we also define approximate covariance function $\widetilde{cov}$ and the approximate cross-covariance function $\widetilde{xcov}$. Here we show the derivation of equations (14) and (15) for stationarized approximate covariance functions (SACF, SAXCF) from the main text, as it is not intuitive. The approximate cross-covariance function $\widetilde{xcov}$ dependent on time t and also on time lag τ is expressed in the main text as:

$$\begin{aligned} \widetilde{xcov}(t, \tau) = & \frac{1}{L_1} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \frac{1}{L_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} f(t - l_1) g(t + \tau - l_1 - l_{12}) dl_{12} dl_1 - \\ & - \frac{1}{L_1} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} f(t - l_1) dl_1 \frac{1}{L_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} g(t + \tau - l_1 - l_{12}) dl_{12} dl_1. \end{aligned} \quad (\text{A.3})$$

Where $f(t)$ and $g(t)$ are two “mother” waveforms, l_1 is random “joint” time shift of both mother waveforms in time, and l_{12} is random relative time shift between the mother waveforms. The time shifts l_1 and l_{12} follow prescribed uniform probability density distributions with defined widths L_1 and L_{12} , respectively.

After substitution of the approximate cross-covariance function in Eq. (A.3) into definition of the stationarized covariance function in Eq. (A.2) the formula for the stationarized approximate cross-covariance reads:

$$\begin{aligned}
\overline{xcov}(\tau) = & \underbrace{\frac{1}{T} \int_{-\infty}^{\infty} \frac{1}{L_1} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \frac{1}{L_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} f(t - l_1) g(t + \tau - l_1 - l_{12}) dl_{12} dl_1 dt}_{\text{I.}} - \\
& - \underbrace{\frac{1}{T} \int_{-\infty}^{\infty} \frac{1}{L_1} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} f(t - l_1) dl_1 \frac{1}{L_1} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \frac{1}{L_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} g(t + \tau - l_1 - l_{12}) dl_{12} dl_1 dt}_{\text{II.}}.
\end{aligned} \tag{A.4}$$

Integrals in Eq. (A.4) can be solved analytically using substitution $z = t - l_1$, and the definition of cross-correlation of real signals $r_{fg}(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t)g(t + \tau)dt$. The first part (I.) of Eq. (A.4) is

$$\begin{aligned}
\text{I.} &= \frac{1}{TL_1L_{12}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} \int_{-\infty}^{\infty} f(z) g(z + \tau - l_{12}) dz dl_{12} dl_1 = \\
&= \frac{1}{TL_1L_{12}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} r_{fg}(\tau - l_{12}) dl_{12} dl_1 = \\
&= \frac{1}{TL_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} r_{fg}(\tau - l_{12}) dl_{12}.
\end{aligned} \tag{A.5}$$

The second part (II.) can be written as

$$\begin{aligned}
\text{II.} &= \frac{1}{TL_1^2L_{12}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} \int_{-\infty}^{\infty} f(z) g(z + \tau - l_{12} + l_1 - k_1) dz dl_{12} dl_1 dk_1 = \\
&= \frac{1}{TL_1^2L_{12}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} r_{fg}(\tau - l_{12} + l_1 - k_1) dl_{12} dl_1 dk_1 =
\end{aligned}$$

... after simple algebra

$$\begin{aligned}
&= \frac{1}{TL_1^2 L_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} \int_{-L_1}^{L_1} (L_1 - |s|) r_{fg}(\tau - l_{12} + s) \, ds \, dl_{12} = \\
&= \frac{1}{TL_{12}} \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} (\wedge_{2L_1}(\tau) * r_{fg}(\tau - l_{12})) \, dl_{12}.
\end{aligned} \tag{A.6}$$

Equation (A.6) is an integral of correlation of the mother waveforms' cross-correlation r_{fg} with the triangle function $\wedge_{2L_1}(\tau)$ with base equal to $2L_1$ and unit area. The triangle function is symmetrical, and hence the result of correlation operation is identical to convolution. Combining of Eqs. (A.4), (A.5) and (A.6) we get simple analytical formula for the stationarized cross-covariance:

$$\begin{aligned}
\overline{xcov}(\tau) &= \frac{1}{TL_{12}} \left[\int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} r_{fg}(\tau - l_{12}) \, dl_{12} - \int_{-\frac{L_{12}}{2}}^{\frac{L_{12}}{2}} (\wedge_{2L_1}(\tau) * r_{fg}(\tau - l_{12})) \, dl_{12} \right] = \\
&= \frac{1}{T} \Pi_{L_{12}}(\tau) * [r_{fg}(\tau) - \wedge_{2L_1}(\tau) * r_{fg}(\tau)],
\end{aligned} \tag{A.7}$$

where $\Pi_{L_{12}}(\tau)$ is the boxcar function with base equal to L_{12} and unit area.

The formula (A.7) describes cross-covariance of randomly shifted signals with uniform probability density distribution of the random time shifts. The cross-correlation r_{fg} depends only on time lag τ , and hence is a characteristics of the joint pattern of the mother waveforms $f(t)$ and $g(t)$. The triangle function $\wedge_{2L_1}(\tau)$ with base equal to $2L_1$ is a characteristics of the possible joint time shifts. And the boxcar function $\Pi_{L_{12}}(\tau)$, with base equal to L_{12} , is characteristics of the possible relative time shifts.

The analytical formula (A.7) describes cross-covariance of two different mother waveforms $f(t)$ and $g(t)$ with generally different time shifts. Such settings is suitable for computing cross-covariance of signals recorded at two different seismic stations or at two components of the same station. The analytical formula further simplifies assuming covariance of the signal with itself¹. The cross-correlation r_{fg} in Eq. (A.7) becomes auto-correlation r_f and $\Pi_{L_{12}}(\tau)$ becomes the Dirac delta function. The stationarized covariance is then

$$\overline{cov}(\tau) = \frac{1}{T} [r_f(\tau) - \wedge_{2L_1}(\tau) * r_f(\tau)]. \tag{A.8}$$

¹i.e. single station, one component ($g(t) = f(t)$; $L_{12} = 0$)

Special cases

Let us discuss some special and limiting cases of Eqs. (A.7) and (A.8) in terms of widths of the uniform probability density distributions L_{12} and L_1 :

- $L_{12} = 0$

The special case when two different mother waveforms $f(t)$ and $g(t)$ have the same time shifts l_1 and no relative time-shift l_{12} . It is the case of cross-covariance of two different components of the seismic receiver². In such case, $\Pi_{L_{12}}(\tau)$ becomes the Dirac delta function $\delta(\tau)$ and Eq (A.7) simplifies to

$$\overline{xcov}(\tau) = \frac{1}{T} [r_{fg}(\tau) - \wedge_{2L_1}(\tau) * r_{fg}(\tau)]. \quad (\text{A.9})$$

- $L_1 = 0$

If the width of the uniform probability density distribution (describing the common time shifts) goes to zero, then the triangle function $\wedge_{2L_1}(\tau)$ becomes the Dirac delta function $\delta(\tau)$. Eq. (A.7) for cross-covariance then goes to zero:

$$\overline{xcov}(\tau) = \frac{1}{T} \Pi_{L_{12}}(\tau) * [r_{fg}(\tau) - \delta(\tau) * r_{fg}(\tau)] = 0 \quad (\text{A.10})$$

This is the case when the two signals are uncorrelated (signals from independent receivers), or if the signals have no time shift and hence are known precisely.

- $L_1 \rightarrow \infty$

The case when L_1 goes to infinity is the limit case of $L_1 \gg T$. Eq. (A.8) then goes to

$$\lim_{L_1 \rightarrow \infty} (\overline{cov}(\tau)) = \frac{1}{T} \left[r_f(\tau) - \frac{1}{L_1} \int_{-L_1}^{L_1} r_f(\tau) d\tau \right] = \frac{r_f(\tau)}{T}. \quad (\text{A.11})$$

This is useful for assessing the maximum possible variance (covariance for $\tau = 0$):

$$\lim_{L_1 \rightarrow \infty} (\overline{cov}(0)) = \frac{r_f(0)}{T} = RMS^2(f), \quad (\text{A.12})$$

where $RMS(f)$ is the root mean square of the mother waveform $f(t)$ over the duration T . Hence, if the width of distribution of time shifts L_1 is much larger than duration of the signal T , then the mean auto-covariance $\widetilde{cov}(0)$ is close to quadratic mean over the whole signal f .

²assuming the same arrival times for the components of the seismic sensor

SUPPLEMENT B: Example of cross-covariance of two nearby receivers

Here we show an example of the cross-covariance of nearby receivers to demonstrate the performance of the simplified analytical formulas compare to Monte-Carlo simulations. The methodology is analogical to auto-covariance presented in the main text. We use the generated 1500 velocity models perturbed by random variations assuming Gaussian distribution characterized by $\sigma_M = 10\%$. Then we simulate GFs (for source-receiver distances 50 and 60km) in the random realizations of the velocity models by discrete wavenumber method (Bouchon, 1981), and we evaluate the cross-covariance of GFs on two different receivers from its definition (Eq. (1) in the main text). The “mother” waveforms for the two stations (evaluated in the mean velocity model) and their 300 realizations in perturbed velocity models are shown in Figs B1a and B1b, for the source-receiver distance 50 km and 60 km, respectively. Cross-covariance matrix estimated using the ensemble of the waveforms (Eq. (1) in the main text) is shown in Fig. B1c. Note that, contrarily to the auto-covariance, the cross-covariance is not symmetrical around the diagonal.

It is in particular the case when two mother waveforms $f(t)$ and $g(t)$ have generally different time shift distributions. However, in the case of nearby stations, the joint time shift distribution has higher influence on cross-covariance than the relative time shift distribution. In other words, the relative inter-station travel path has smaller effect on the GFs than the shared source-receive travel path ($L_1 > L_{12} > 0$). The distributions of joint and relative time shifts l_1 and l_{12} for this example are shown in Figs B2a and B2b, respectively. Fig. B1d displays the approximated cross-covariance matrix (AXCF) obtained using Eq. (11) from the main text. There is some difference between the Monte-Carlo XCF and AXCF cross-covariance matrices, but they are still close in terms of pattern including signs. We note that the difference might increase with mutual distance of receivers, as the joint time shift $L_1 \rightarrow 0$ (reduction of the joint GFs uncertainty). Finally, stationarized Monte-Carlo and approximate (SAXCF) cross-covariances, compared in Fig. B1f, agree very well. Note that the SAXCF is not centered around zero time lag due to the different arrival times on the stations, as captured by the cross-correlation r_{fg} .

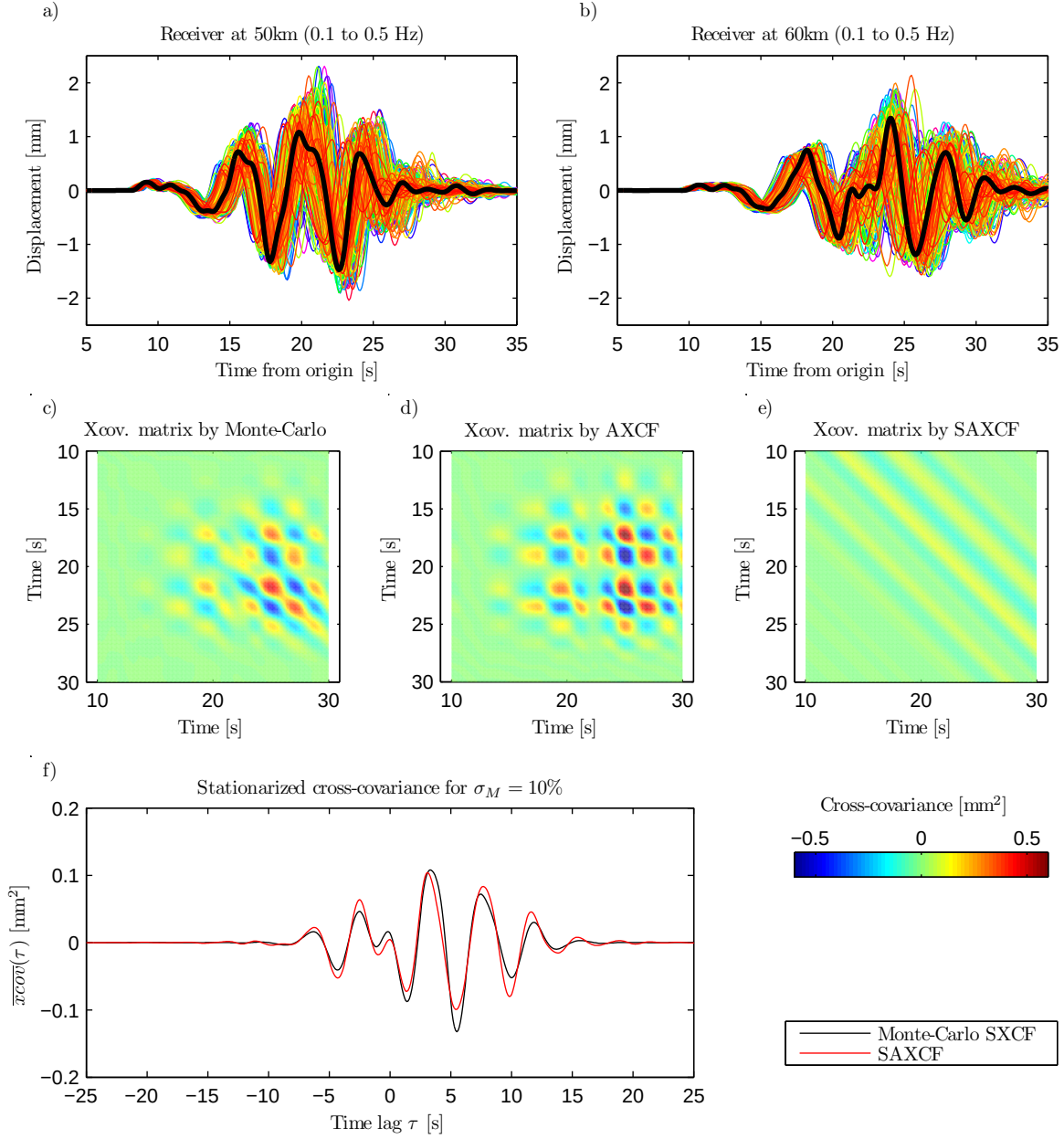


Figure B1: An example of the cross-covariance for two nearby receivers. Panels **a)** and **b)** show two sets of GFs for the source-receiver distances 50 km and 60 km, respectively, computed by the discrete wavenumber method ($\sigma_M = 10\%$). The black lines are the mother GFs calculated in the mean velocity model. Other colors of waveforms have no meaning and are used only for clearer view. Panel **c)** show full cross-covariance matrices for velocity model perturbations ($\sigma_M = 10\%$), as obtained by the Monte-Carlo approach, while matrices in panels **d)** and **e)** were obtained by the approximate formulas in Eq. (11) and Eq. (15), respectively. Duration of the dominant part of the earthquake signal was set up as $T = 15[s]$. Finally, panel **f)** shows comparison of stationarized Monte-Carlo cross-covariance function (black) and SAXCF obtained using Eq. (15) (red).

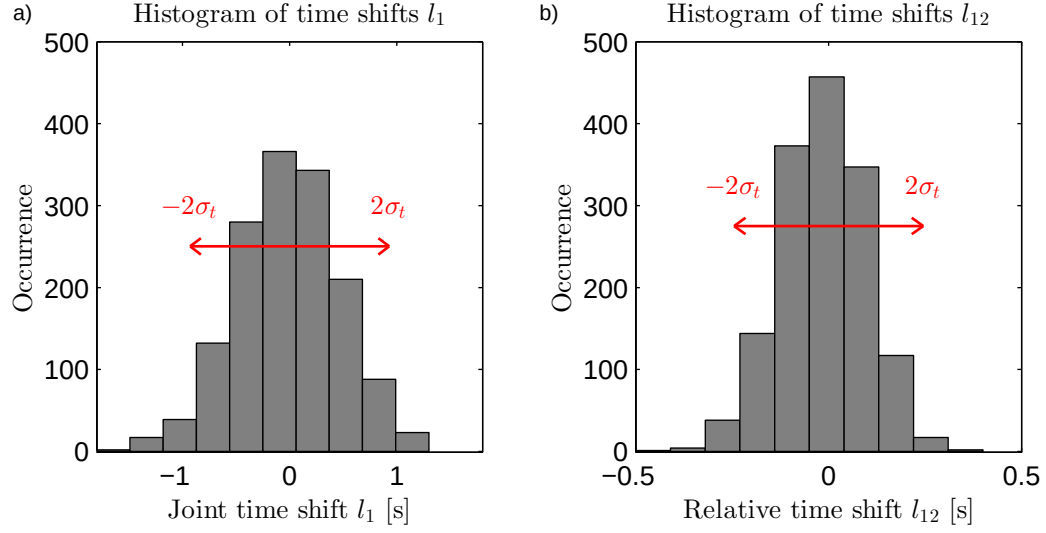


Figure B2: *Distributions of time shifts of GFs due to velocity model perturbations for the example in Fig. B1. Time shifts are estimated from waveforms filtered at very low frequencies (0.05-0.09 Hz) to avoid cycle skipping. The red lines show the characteristic widths of the time shift distributions ($4\sigma_t$): **a)** Joint time shifts, $4\sigma_t = 1.86[s]$; **b)** Relative (inter-station) time shifts, $4\sigma_t = 0.49[s]$.*

SUPPLEMENT C: Examples of complete covariance matrices used in the source inversion

In the main text we perform synthetic source inversion tests considering various covariance matrices. A complete covariance matrix of such inverse problem has block structure. Auto-covariance matrices (diagonal, SACF or ACF) for the individual waveforms are arranged along the diagonal. Off-diagonal blocks then contain cross-covariance matrices (SAXCF or AXCF) among the waveforms. We consider only cross-covariances between components of each receiver, neglecting the inter-station cross-covariances. Neglected blocks then have zero values. As an example, we show here only submatrix corresponding to the DSF station, as the complete covariance matrix is too large for the presentation purpose. The synthetic 3-component waveforms for the DSF station are shown in Fig. C1. The covariance submatrices are then shown in Fig. C2 for the six different types of covariance matrices discussed in the main text. Note especially the effect of the stronger signal on the North component, and the effect of the taper on the SACF (and SAXCF) matrices (see main text for the taper).

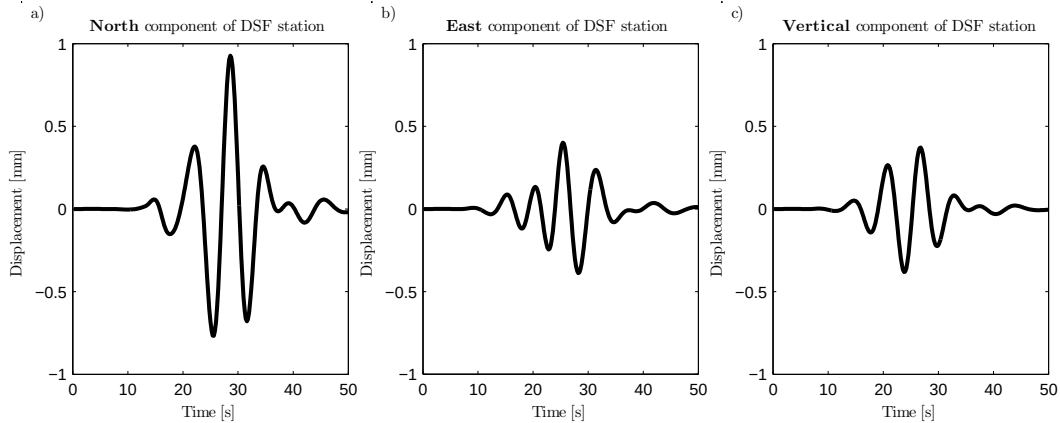


Figure C1: *Synthetic waveforms on the DSF station for the synthetic inversion test described in the main text. Data are filtered by Butterworth bandpass filter between 0.1 to 0.2 Hz.*

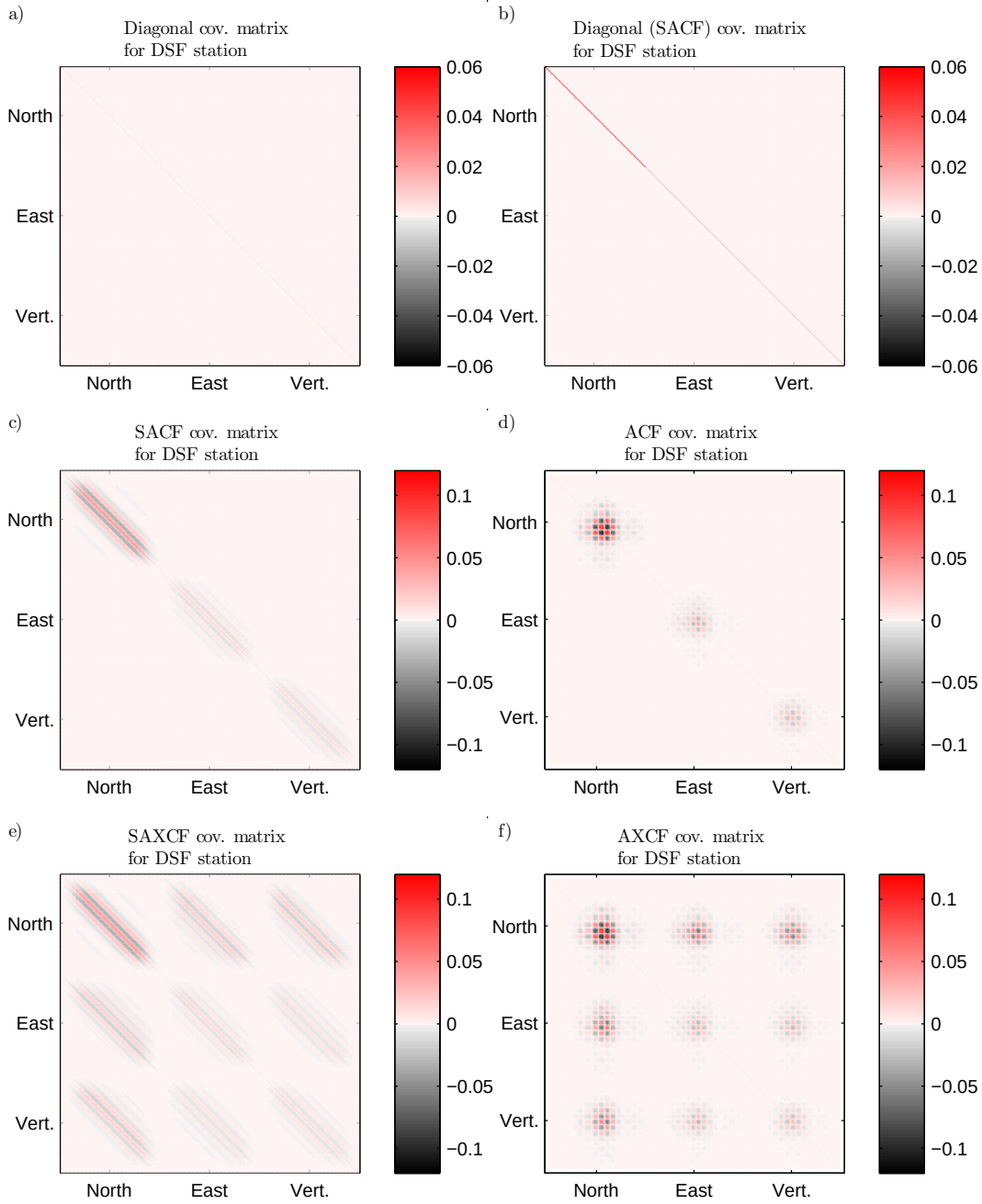


Figure C2: Submatrices for the six different types of covariance matrices discussed in the main text (see titles in figure) corresponding to the DSF station. There are shown 3 blocks of auto-covariances and 6 blocks of cross-covariances among DSF station's components in each submatrix. The values shown by color have units $[mm^2]$.

SUPPLEMENT D: Fit of the standardized waveforms for the real data inversion example

The inversion used in the main text minimizes the difference (waveform residuals) between the measured data and synthetics by means of L2 norm. In this section we show example of the best fit of the measured data and synthetics for the real event from 25 April 2012 from Corinth Gulf on all stations and components. If the covariance matrix is used, the inversion minimizes the standardized residuals instead of waveform residuals. Where the standardized data are waveforms multiplied by the triangular matrix from the Cholesky decomposition (see Eq. (23) in the main text). In Fig. D1 we show the best fitting solution of the inversion of real event with the constant diagonal covariance matrix. Normalized displacement (black) and synthetic displacement (red) are shown on all stations and components used in the inversion. In Figs D2 and D3 we show examples of standardized displacement (black) and synthetics (color) standardized by various types of covariance matrices (ACF and SACF respectively).

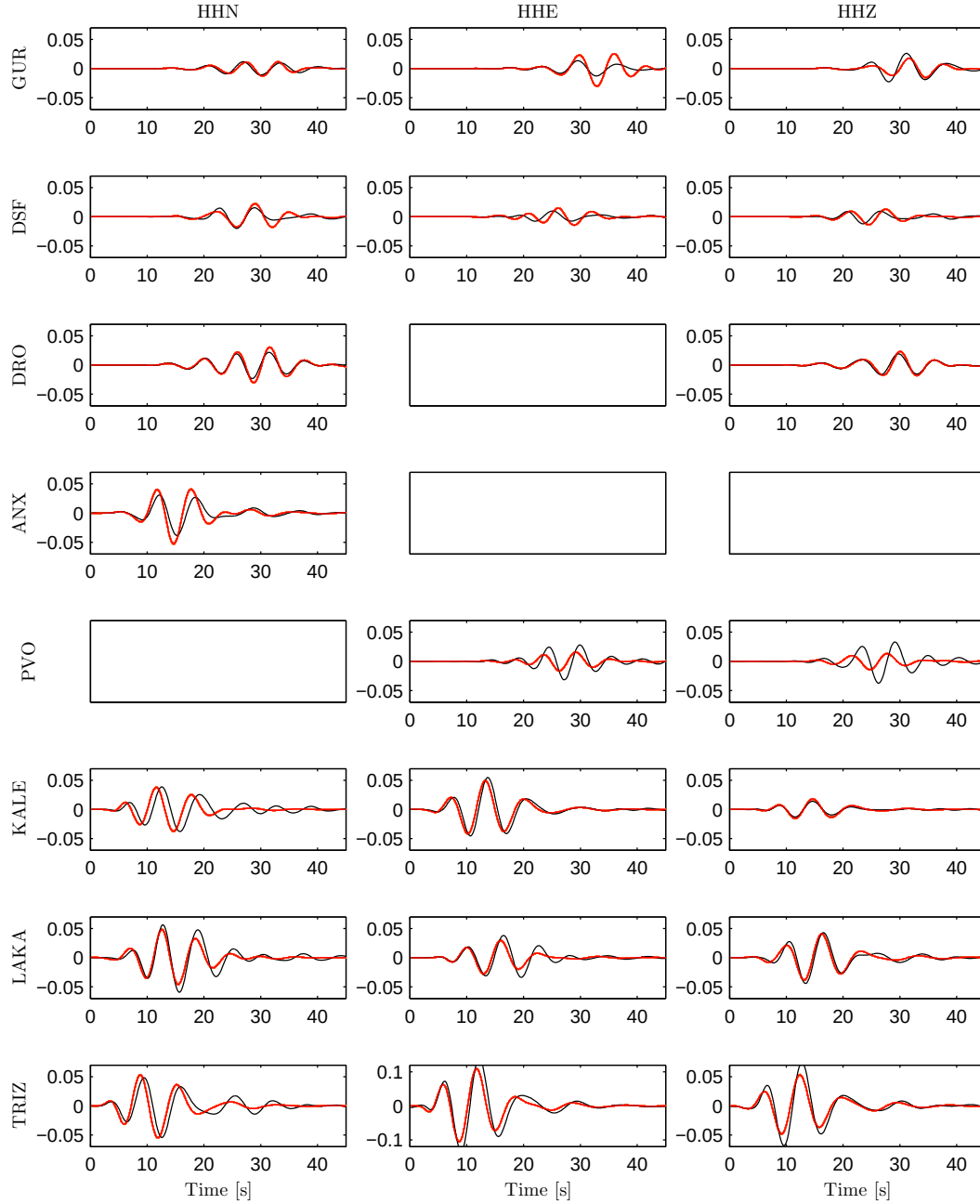


Figure D1: *Normalized displacement (black) and synthetic (color) disregarding uncertainty of GFs. Data are corrected by instrument response and filtered by Butterworth bandpass filter between 0.1 to 0.2 Hz. Black waveforms are normalized displacement and colored waveforms are 300 synthetic realizations of normalized displacement perturbed around the best fitting solution by posteriori covariance.*

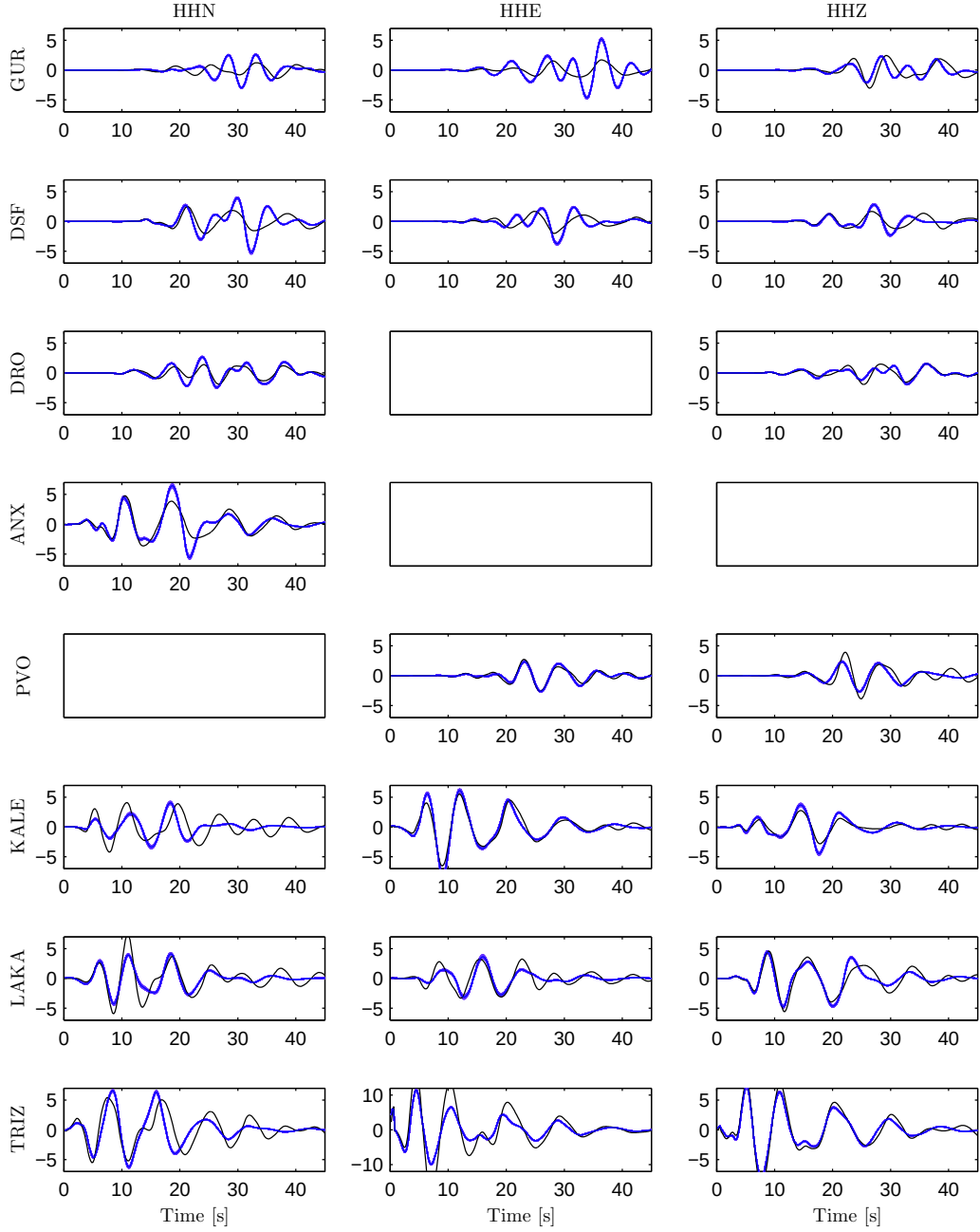


Figure D2: *Same as Fig. D1, but for standardized displacement and synthetic realizations considering uncertainty of GFs by means of ACF.*

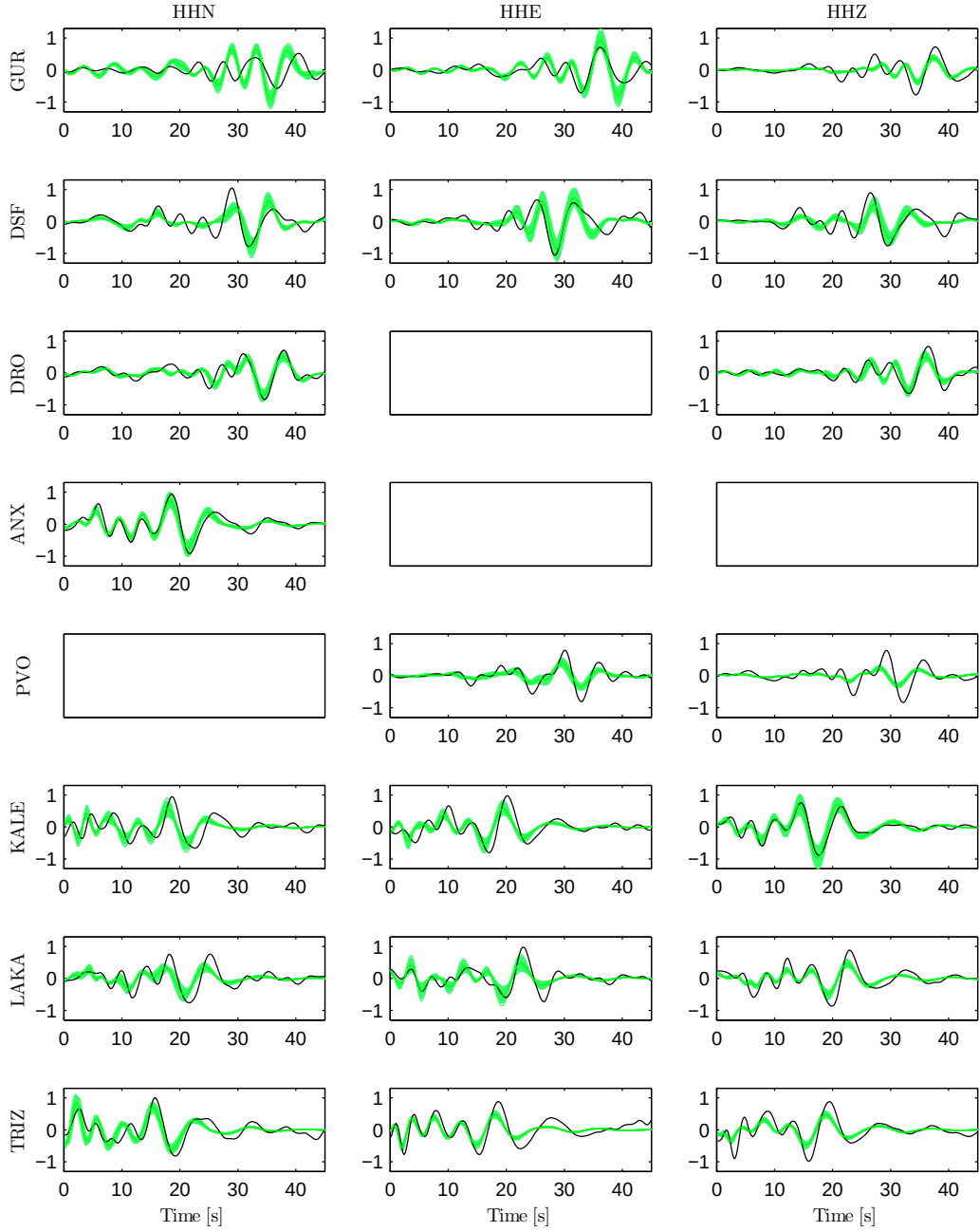


Figure D3: *Same as Fig. D1, but for standardized displacement and synthetic realizations considering uncertainty of GFs by means of SACF.*