Lecture course on

Sea level variations and global geodynamics

2. Rheology and Loading Problems

(November 8, start 12:20)

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November 7–11, 2011

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Outline

1). Basic things about Rheology

2). “Normal modes” (Spectra, Love numbers...)

3). “Loading problems” (recent PGR benchmark activity)

4). Special topic: non-traditional computation of Love numbers

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Charles University in Prague, Department of Geophysics

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1). Basic things about Rheology

"Rheology is the study of the flow of matter"

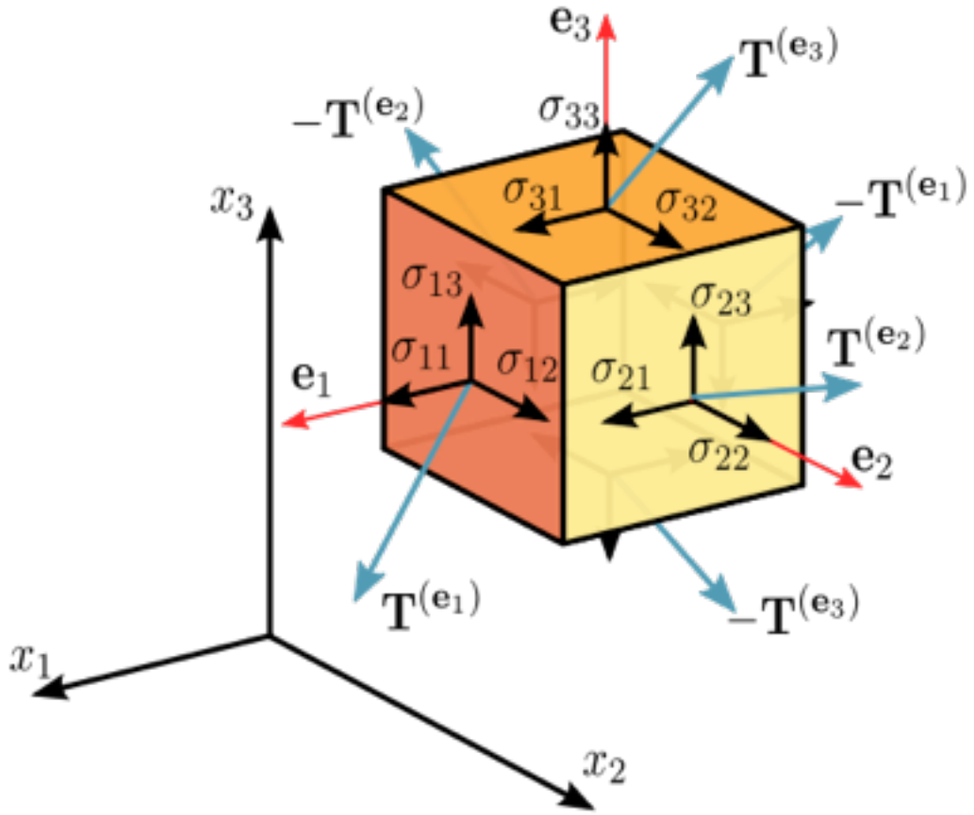
The term was coined by Eugene C. Bingham in 1920, from a suggestion by Markus Reiner and inspired by the quotation mistakenly attributed to Heraclitus, (actually from Simplicius) **panta rei**, "everything flows". *(from wiki)*

It is a fundamental discipline in Geophysics, since it allows to describe the **flow of solid or fluid Earth materials.**

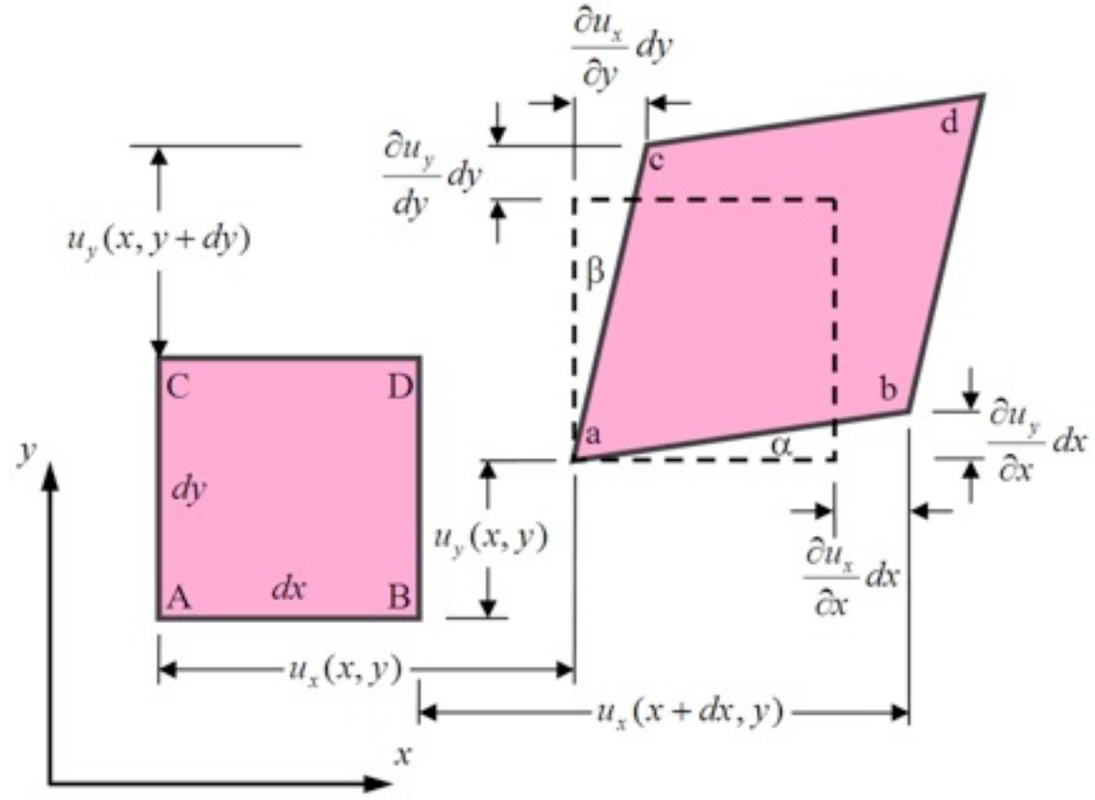
But it is also important in **chemistry**, **material sciences**, **biology, medicine** and other disciplines...

We will look at *phenomenological* aspects of RHEOLOGY (as opposite to microphysical rheology)

STRESS



STRAIN



$$\sigma = \frac{\text{Force}}{\text{unit area}}$$

$$\epsilon = \frac{\text{Elongation}}{\text{Initial length}}$$

The relation between stress and strain is expressed by **constitutive equations**

$F = -kx$	Linear elasticity *(Spring law)*
$F_{friction} = \mu N$	Friction *(Amonton's law)*
$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$	Elasticity *(Hooke's law, 3D)*
$\sigma = C\epsilon$	Elasticity *(Hooke's law, 1D)*
$\overrightarrow{J} = -\mathbf{D}\overrightarrow{\nabla c}$	Diffusion *(Fick's law)*
$q = -\frac{k}{\mu}\Delta P$	Flow in porous media *(Darcy's law)*
$V = RI$	Electrical circuits *(Ohm's law)*

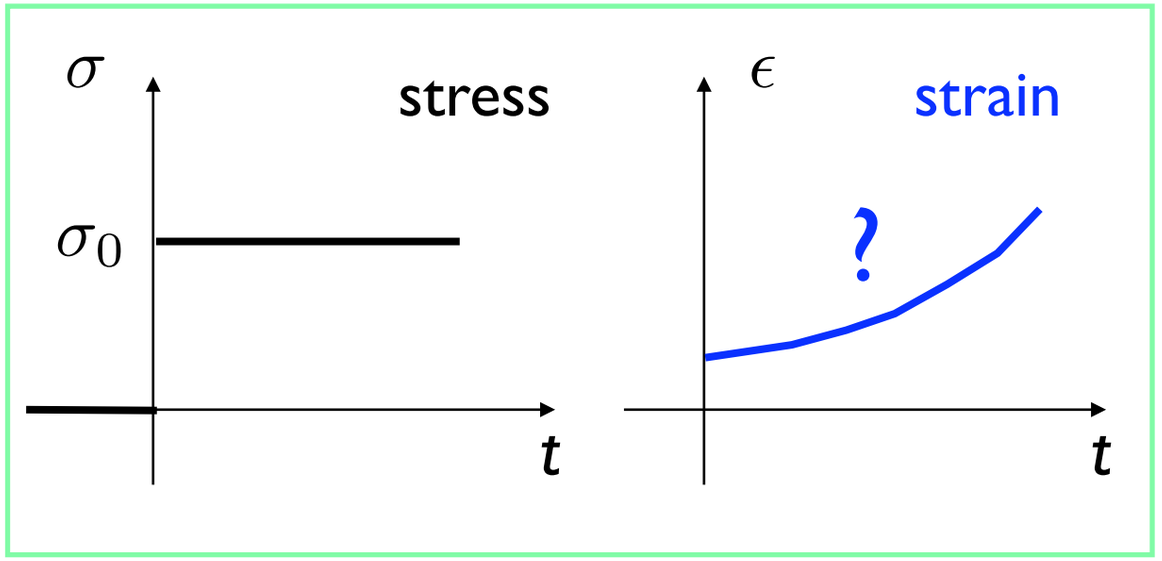
Mechanical experiments



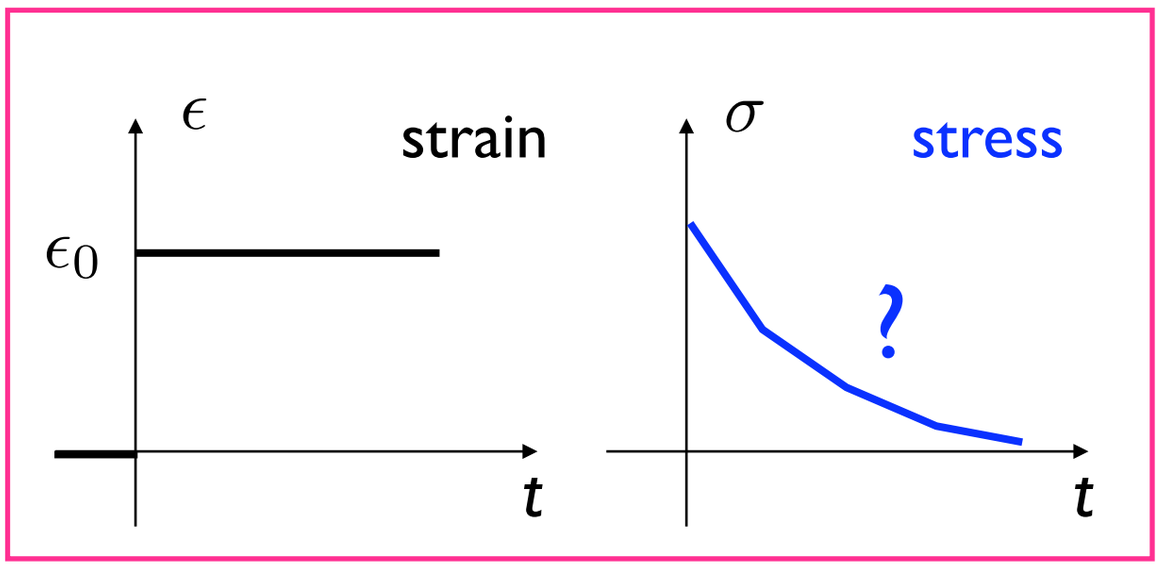
$\sigma(t)$ $\epsilon(t)$

$\epsilon(t)$ $\sigma(t)$

CREEP *EXPERIMENT*



INPUT OUTPUT



RELAXATION *EXPERIMENT*

CREEP EXPERIMENTS are conceptually very useful since they help to classify the rhological behavior:

Liquid and **solid** characters are relevant at “long times”:

- if the material flows indefinitely, it is considered a liquid,
- if, by contrast, the material after some deformation, eventually resists, it is considered a solid.

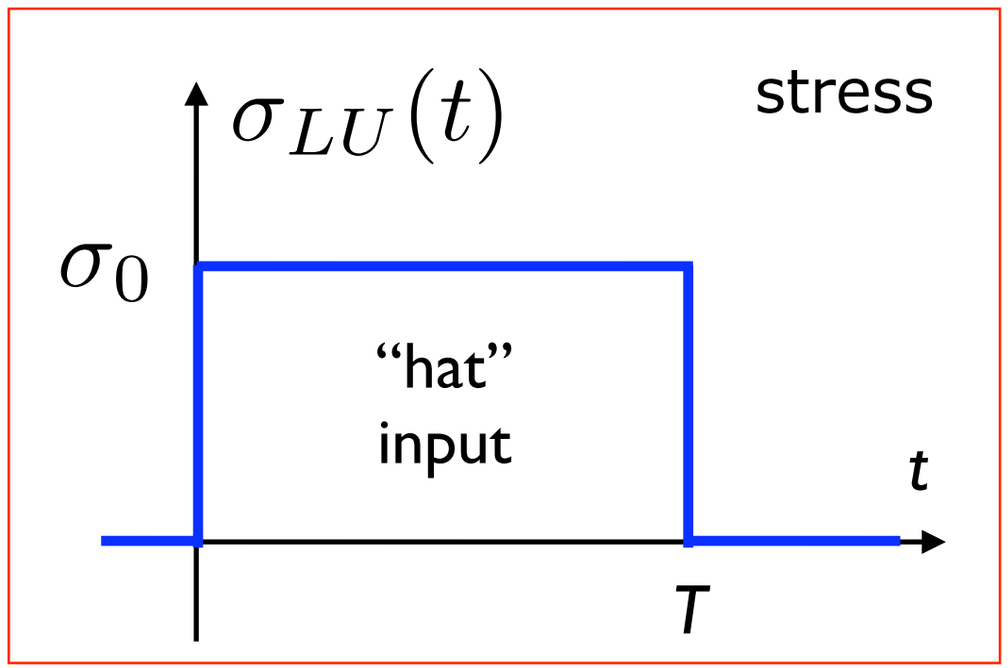
Elasticity, Newtonian behavior and visco-elasticity are relevant at “laboratory times”:

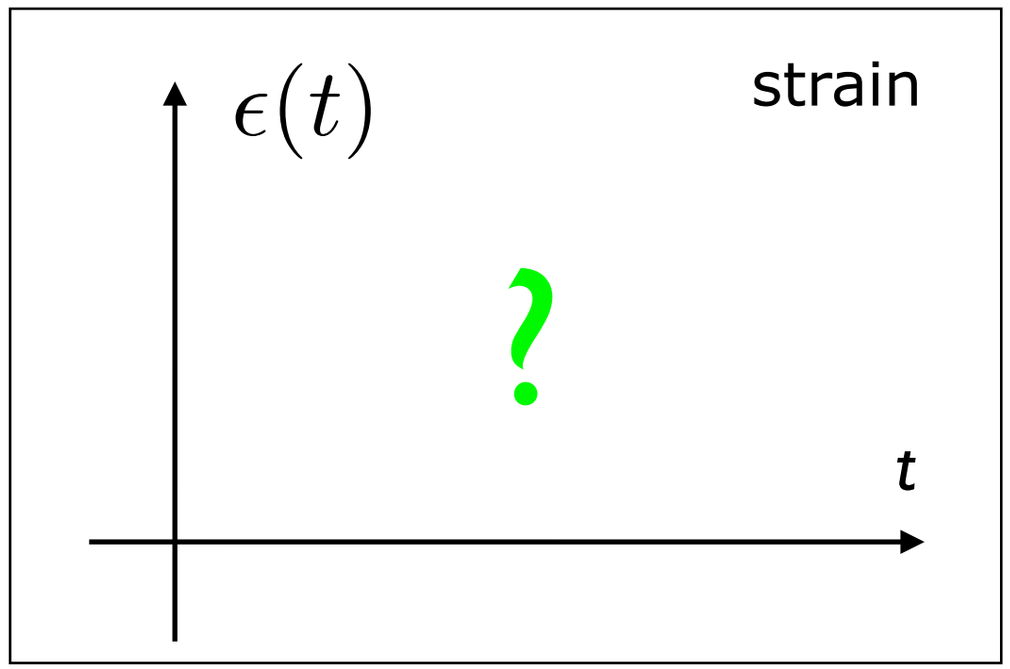
- if the stress is proportional lo strain, the body is “elastic”,
- if the stress is proportional to strain rate, it is “newtonian”,
- if none of the above applies, the body is “visco-elastic”.

LOADING / UNLOADING experiment

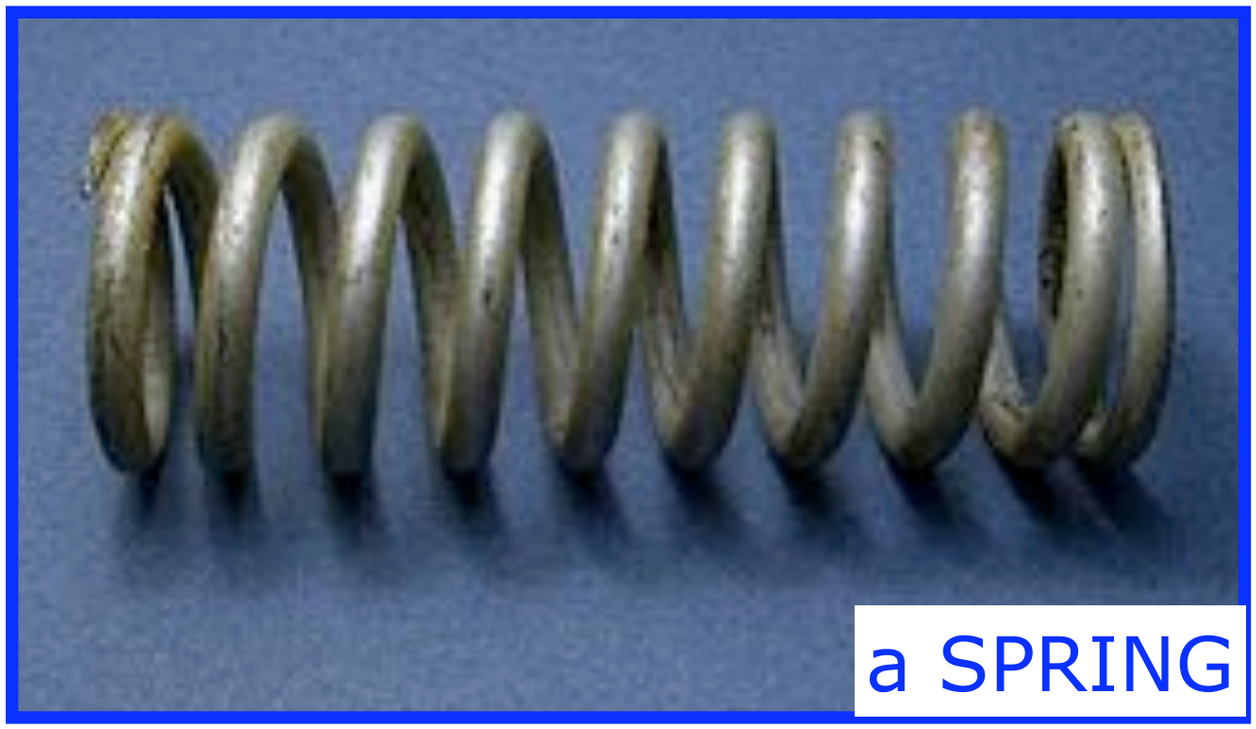
The input stress follows an "Hat function", of duration *T*:

$$\sigma_{LU}(t) = \sigma_0[H(t) - H(t-T)], \quad T > 0$$





$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



The SPRING mimics ELASTICITY:

$$F = -k(x - x_0)$$

k = "spring constant"

ELASTIC behavior is often represented by a **"mechanical analogue",** also useful for general *mechanical systems.*



"Elastic element" (a SOLID)

Sir Robert Hooke
(1635 – 1703)



The fundamental law of Elasticity is:

"STRESS is proportional to *STRAIN*".

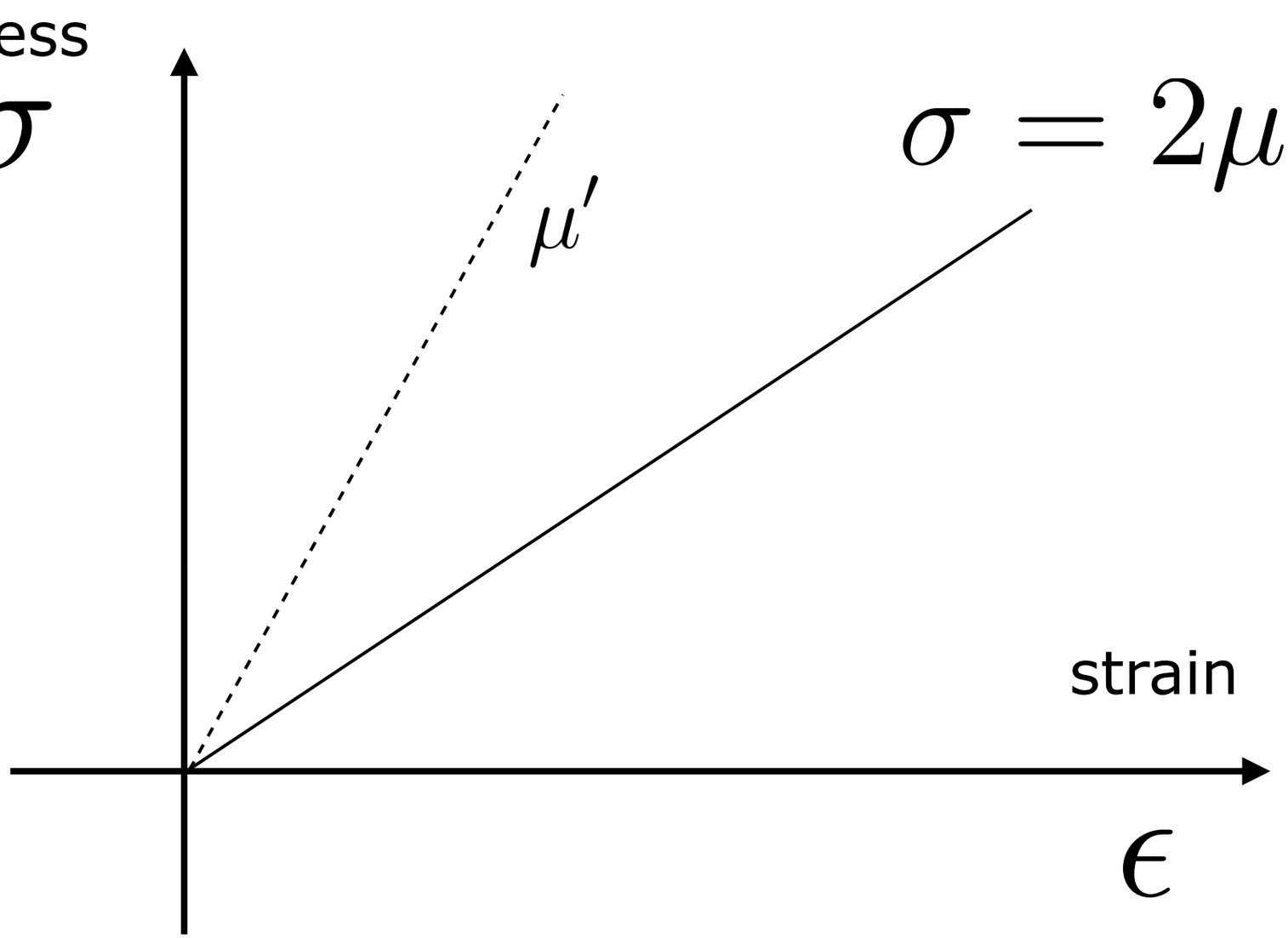
In formulas:

$$\frac{\sigma}{\epsilon} = \text{constant}$$

Hence:

$$\sigma = 2\mu\epsilon$$

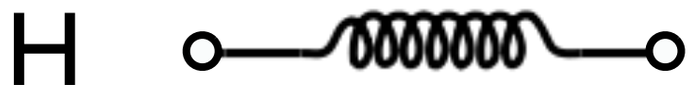
where μ is a "material constant" known as "rigidity", or "shear modulus" (having the dimensions of a stress).


stress
σ
μ'
$\sigma = 2\mu\epsilon$
strain
ϵ

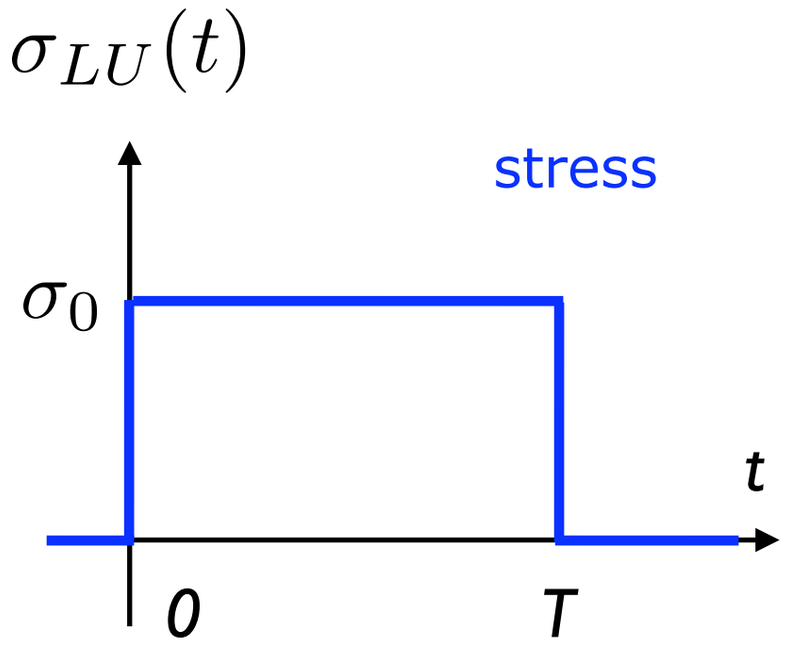


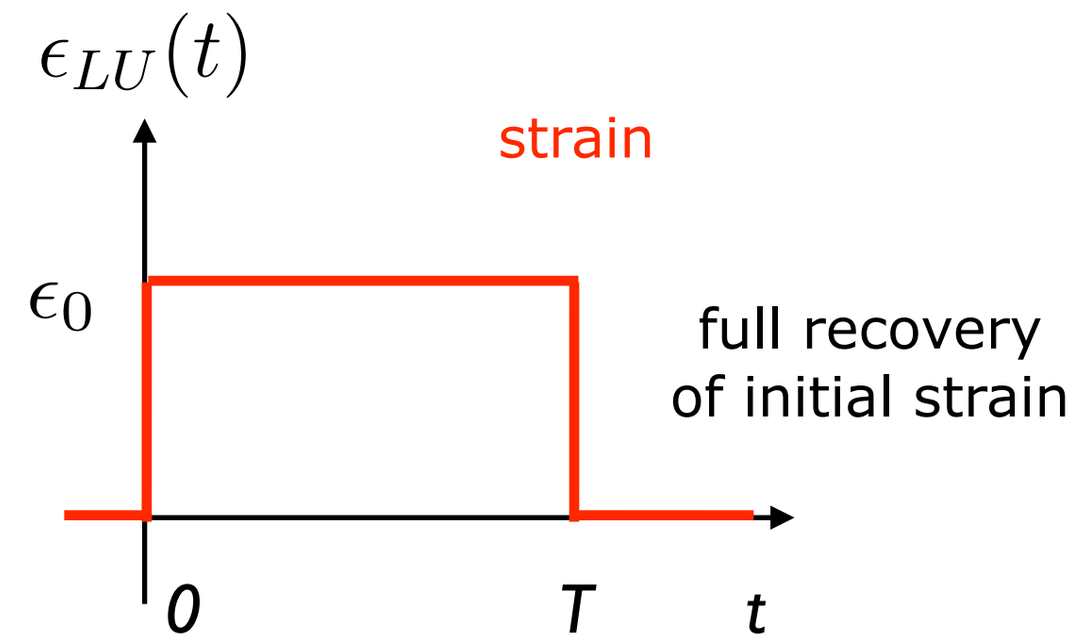
Hooke body

$$\sigma_{LU}(t) = \sigma_0[H(t) - H(t-T)]$$



$$\epsilon_{LU}(t) = \epsilon_0[H(t) - H(t-T)]$$

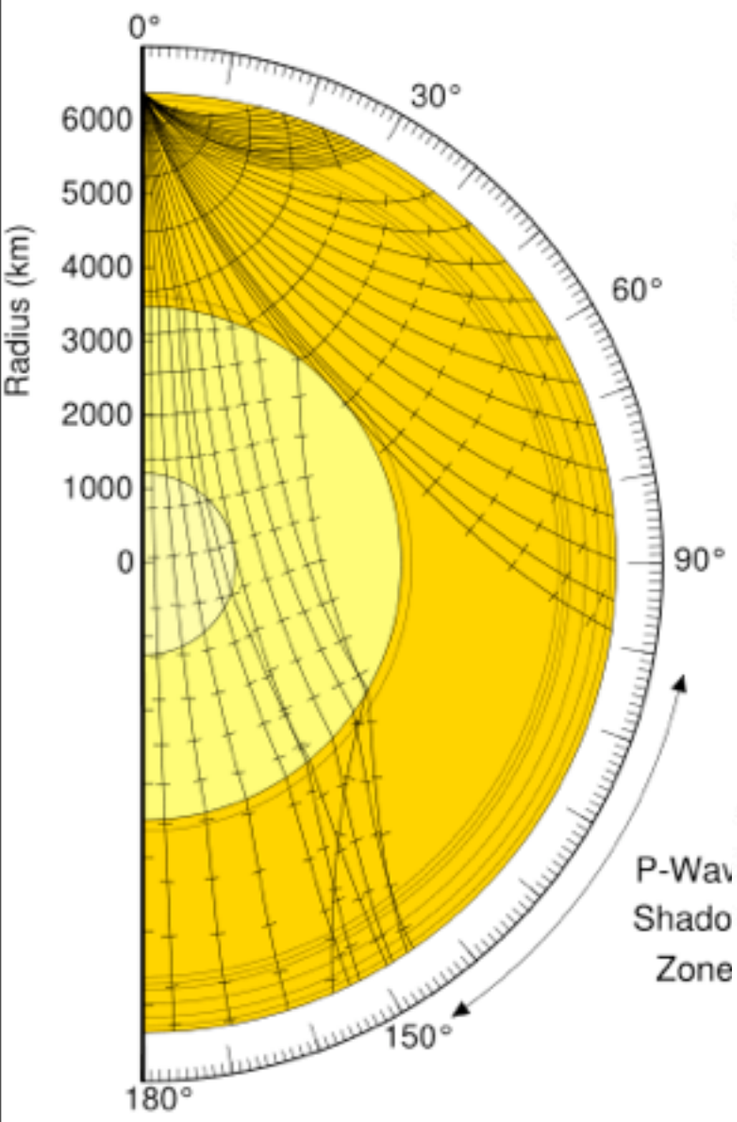


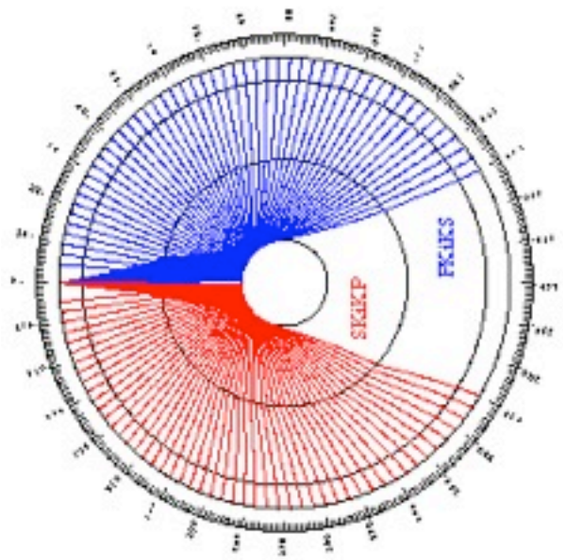


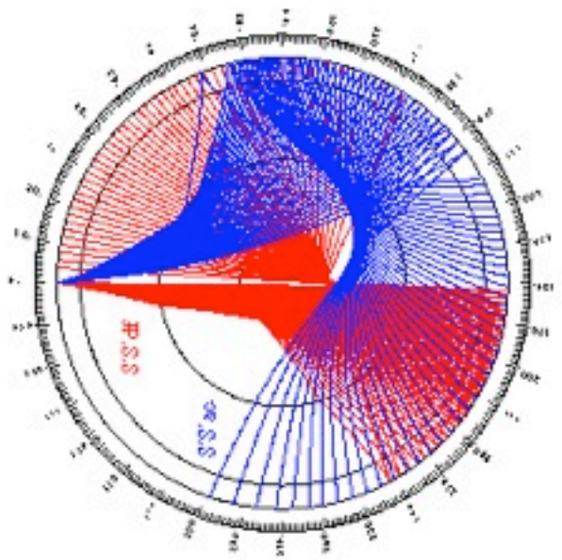
A “FULL RECOVERY” of initial strain characterizes the Hooke body.

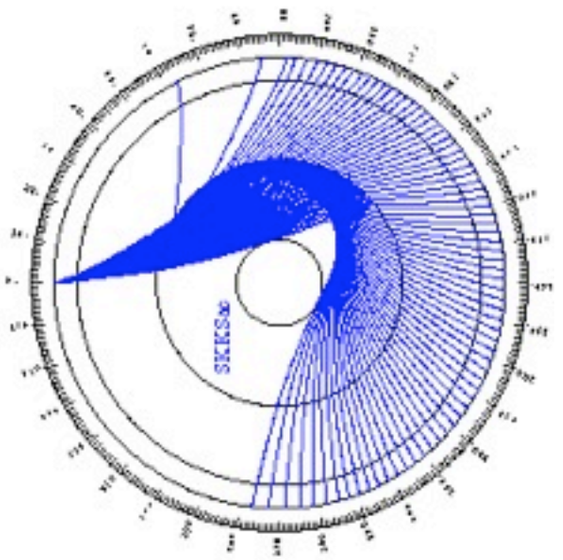
"Wave equation"

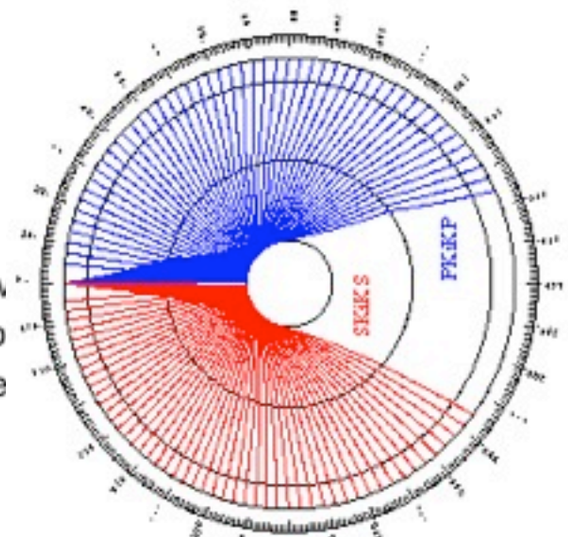
$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (K + \frac{4}{3}\mu)\vec{\nabla}(\vec{\nabla} \cdot u) - \mu\vec{\nabla} \times (\vec{\nabla} \times \vec{u})$$

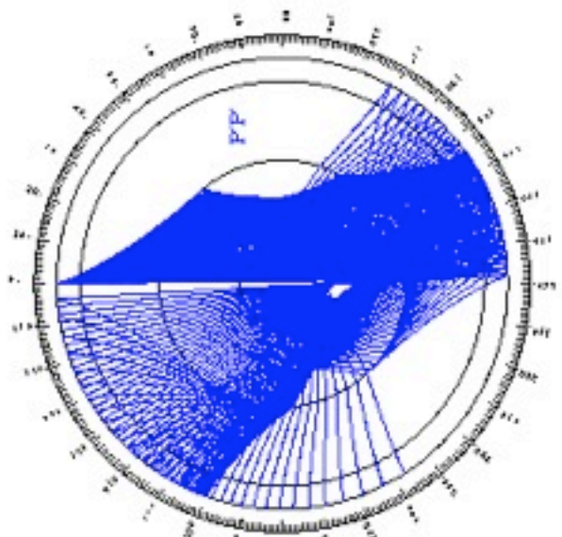


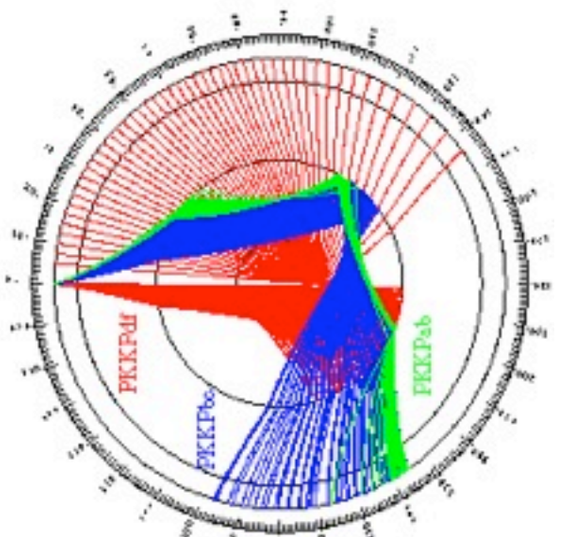


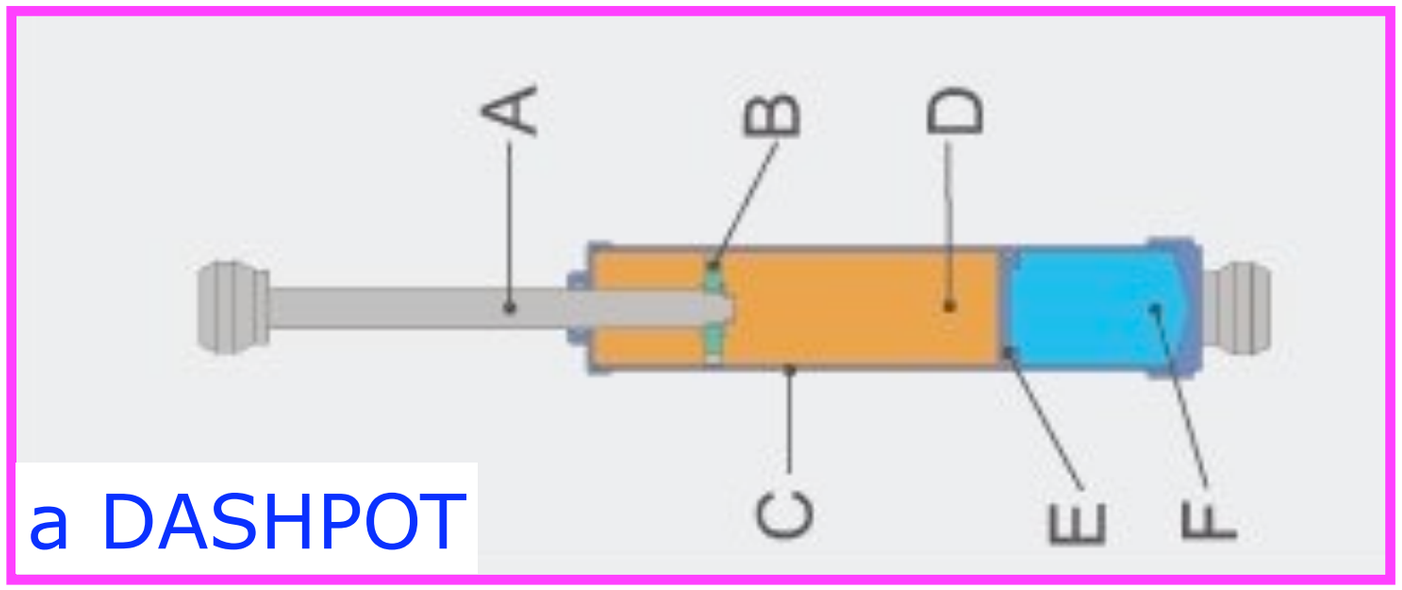












a DASHPOT

a DAMPER

The DAMPER mimics VISCOUS behavior:

$$F = -k_v \frac{dx}{dt}$$

k_v = dissipation constant

VISCOUS behavior is also represented by a **"mechanical analogue",** as for general mechanical systems.



"Newtonian element"
(a FLUID)



Sir Isaac Newton,
(4 January 1643 – 31 March 1727)

The fundamental Newton law is:

"STRESS is proportional to STRAIN *RATE*".

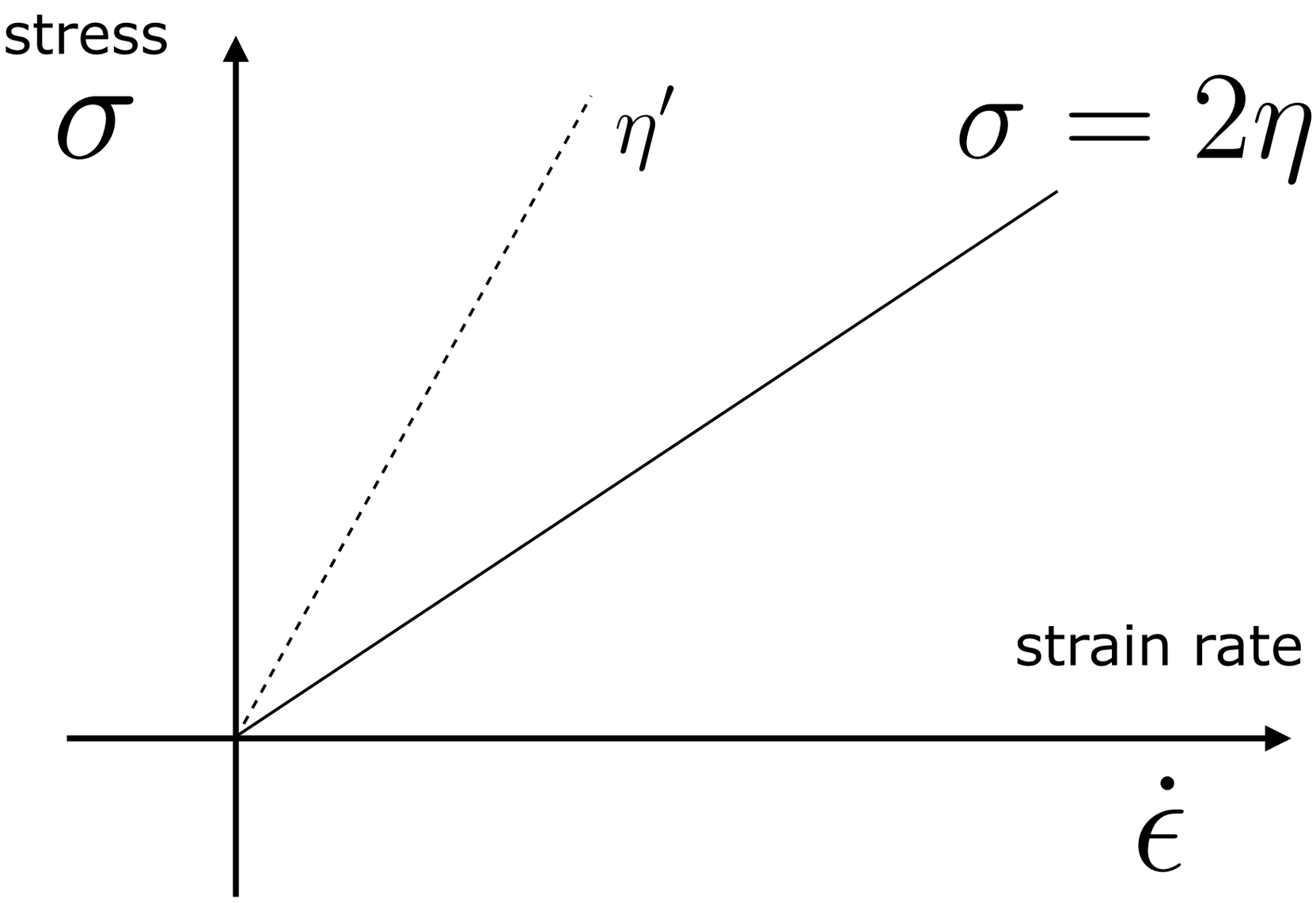
In formulas:

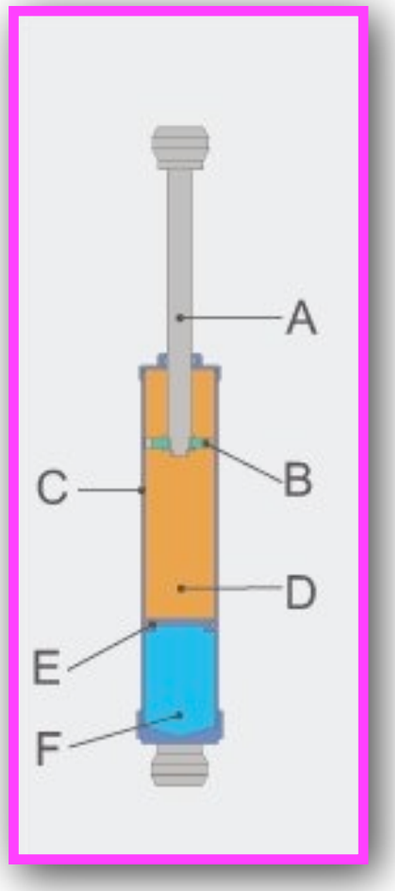
$$\frac{\sigma}{\dot{\epsilon}} = \text{constant}$$

Hence:

$$\sigma = 2\eta\dot{\epsilon}$$

where η is a "material constant" (not dependent on stress, but possibly on P and T), known as "viscosity".

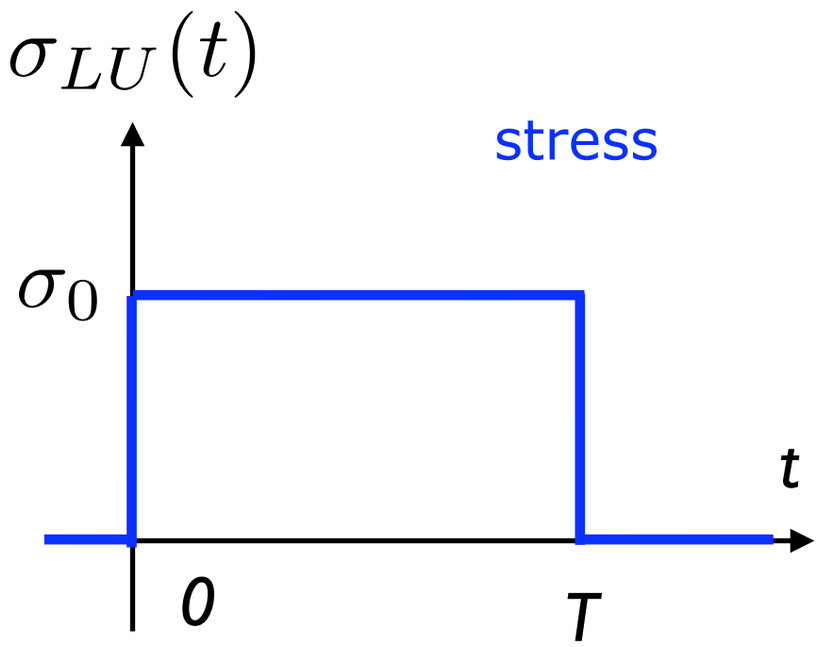

stress
σ
η'
$\sigma = 2\eta\dot{\epsilon}$
strain rate
$\dot{\epsilon}$


A
B
C
D
E
F

Newton body

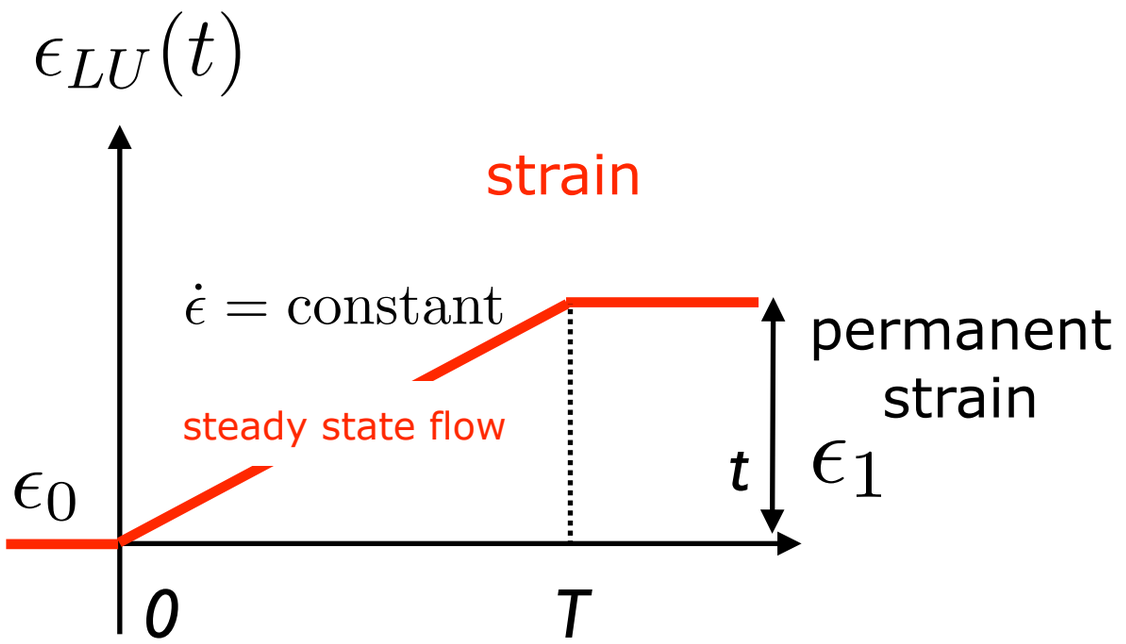
LOADING / UNLOADING experiment

$$\sigma_{LU}(t) = \sigma_0[H(t) - H(t-T)]$$





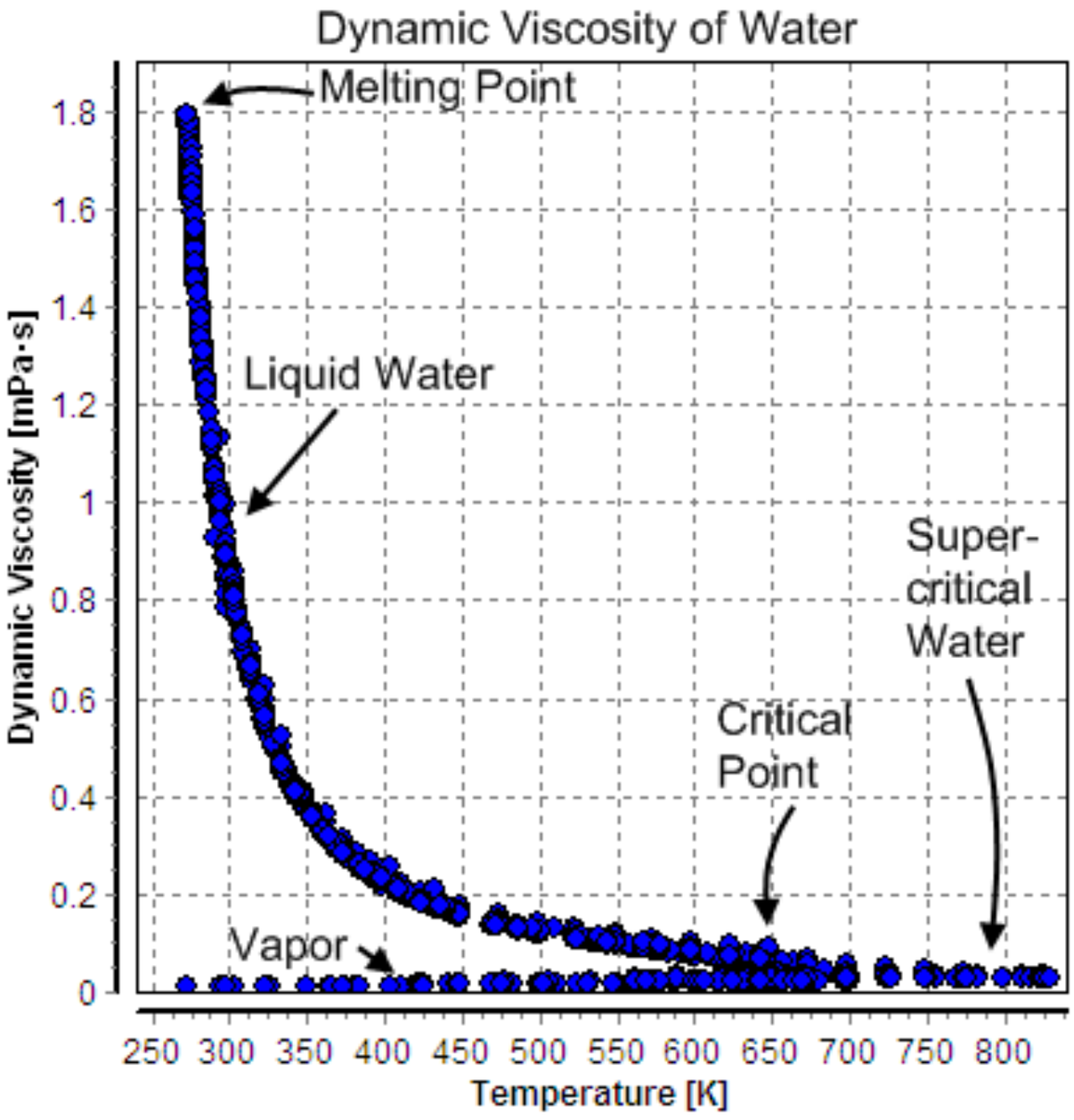
$$\epsilon_{LU}(t) = \frac{\sigma_0}{2\eta}[t - (t-T)H(t-T)]$$



Lack of *recovery* of initial strain and permenent strain (liquid behavior).

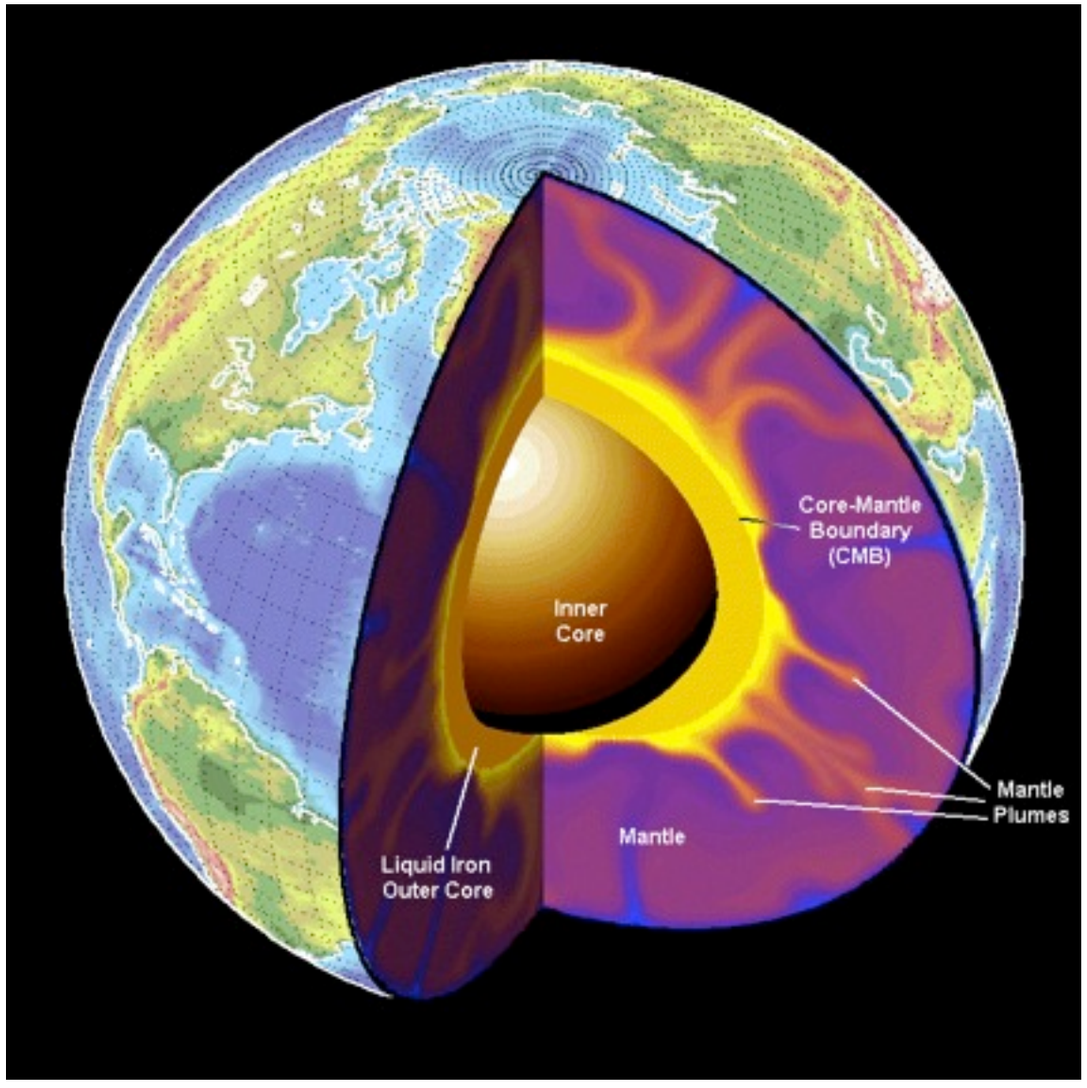
Steady state flow observed until time t=T.

WATER

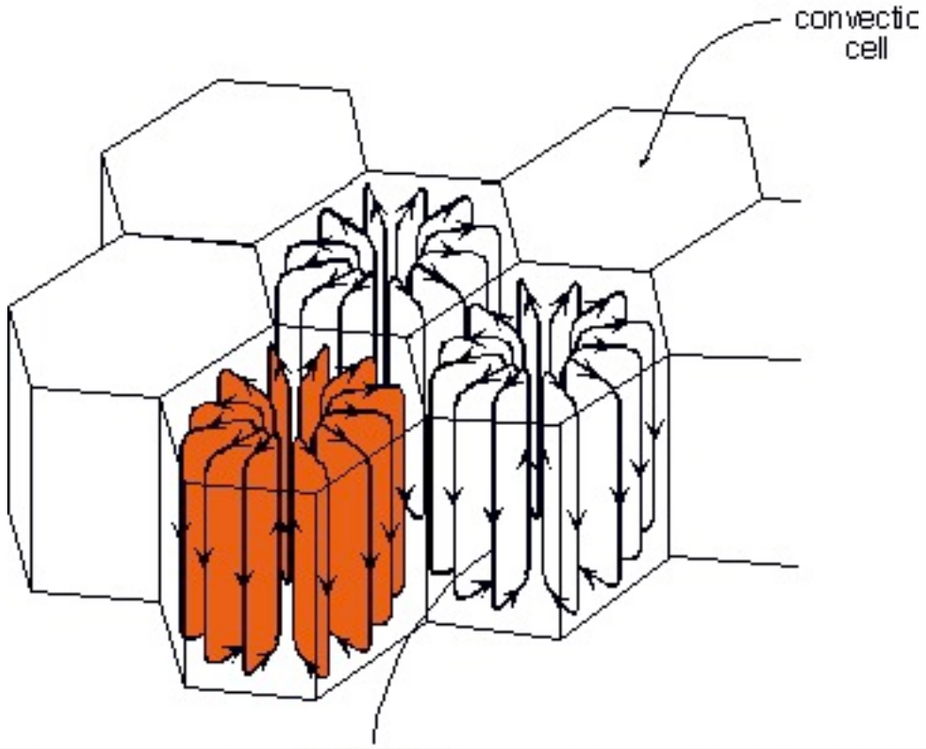


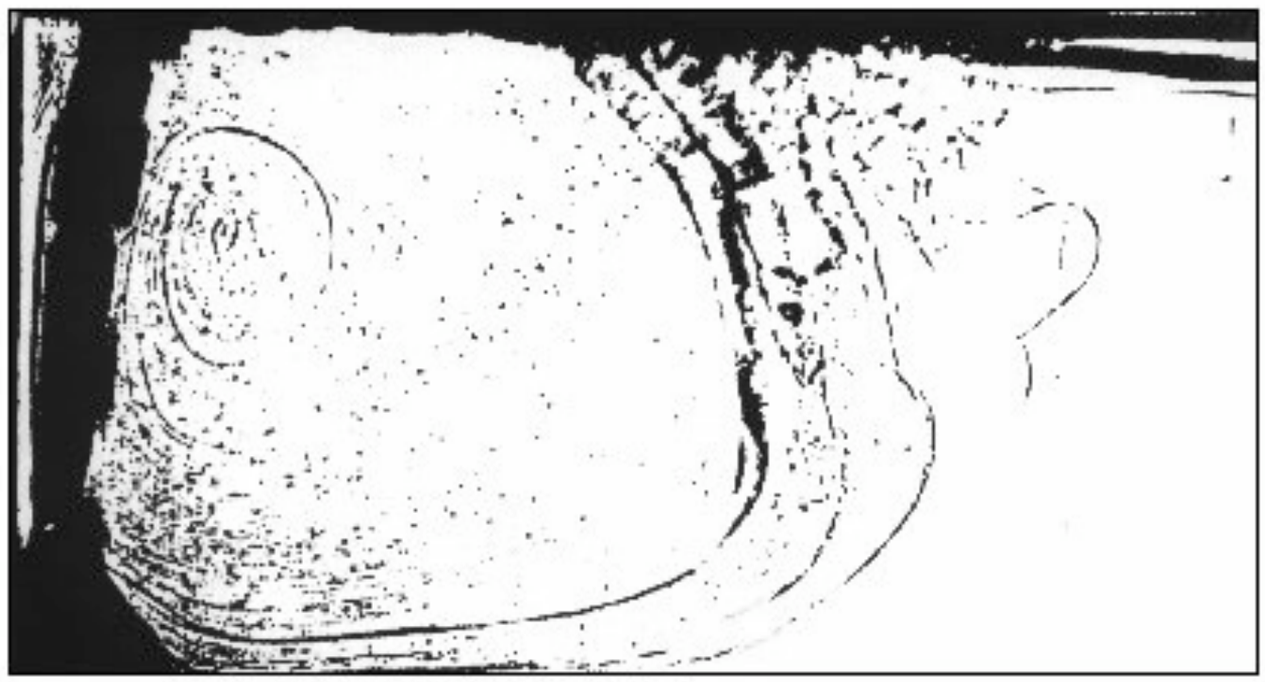
$$\eta \approx 10^{-3} Pa \cdot s$$

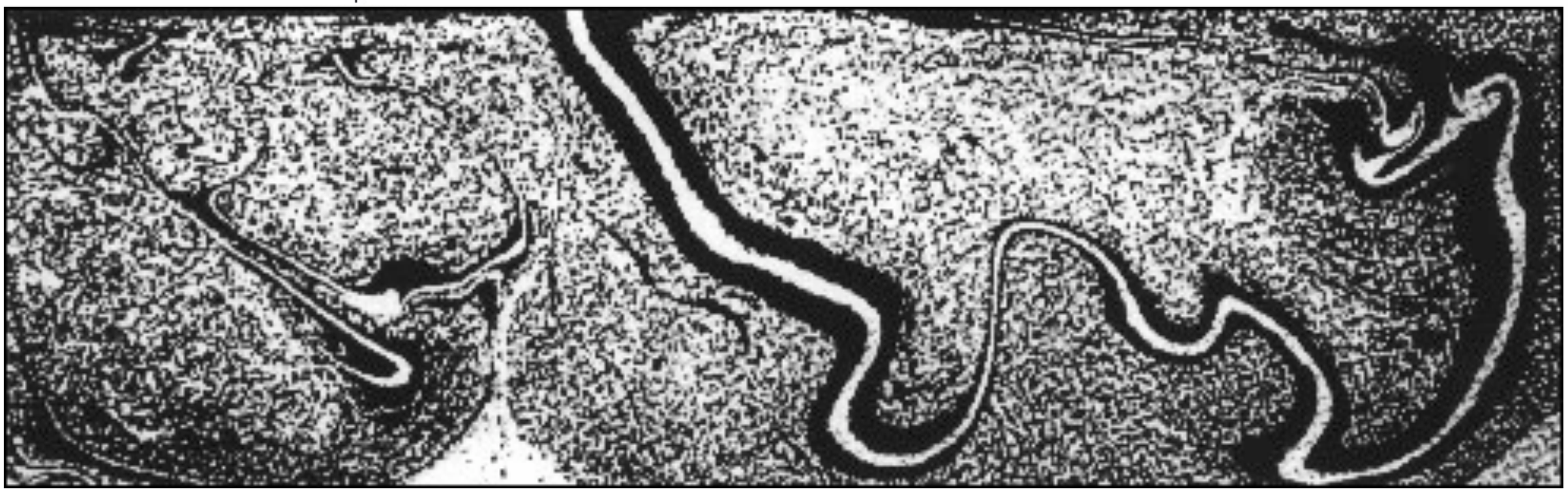
EARTH



$$\eta \approx 10^{21} Pa \cdot s$$







Experimental mantle convection

4). Linear viscoelasticity



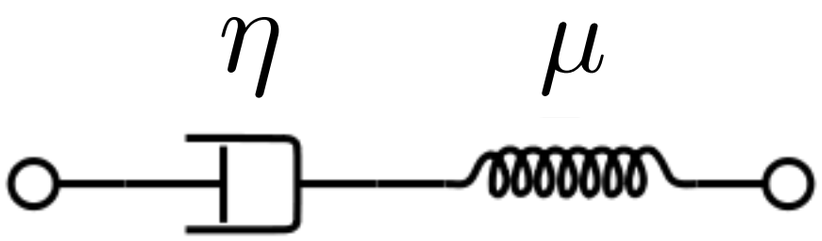
James Clerk Maxwell (13 June 1831 – 5 November 1879) was a Scottish mathematician and theoretical physicist.

His most significant achievement was formulating a set of equations — named **Maxwell's equations** — that for the first time expressed the basic laws of electricity and magnetism in a *unified fashion.*

*James Clerk Maxwell proposed the **"Maxwell" rheology** in 1867.*

A Maxwell material is a viscoelastic material having the properties **both** of elasticity and viscosity. It is also known as a Maxwell solid (*)

The Maxwell model can be represented by a purely viscous damper and a purely elastic spring connected **consecutively**, ("in series") as shown here:



Machanical analogue of the Maxwell body:
a *damper* in series with a *spring*.

Using Reiner's notation:

M = N - H

(*) BUT it is in fact a fluid, according to our definitions !!!

In the Maxwell model, the spring and the dashpot are subject to the ***same stress***, but the strain is different for the two elements.

The TOTAL strain is:

$$\epsilon_{tot} \equiv \epsilon = \epsilon_H + \epsilon_N$$

where, from above:

$$\epsilon_H = \frac{\sigma}{2\mu} \quad \text{and} \quad \dot{\epsilon}_N = \frac{\sigma}{2\eta}$$

that gives Maxwell rheological equation:

$$\dot{\epsilon} = \frac{\dot{\sigma}}{2\mu} + \frac{\sigma}{2\eta}$$

In a **creep test**, the system is subject to a constant *stress*:

$$\sigma(t) = \sigma_0, \quad t \geq 0$$

Hence, solving Maxwell equation gives:

$$\epsilon(t) = \epsilon_0 + \frac{\sigma_0}{2\eta}\, t$$

where ϵ_0 is strain at time *t=0*:

$$\epsilon_0 = \frac{\sigma_0}{2\mu}$$

Hence, Maxwell body under a sudden stress deforms elastically, and subsequently it flows at a constant rate.

In a **relaxation test**, the system is subject to a constant *strain*:

$$\epsilon(t) = \epsilon_0, \quad t \geq 0$$

Hence, solving Maxwell equation gives:

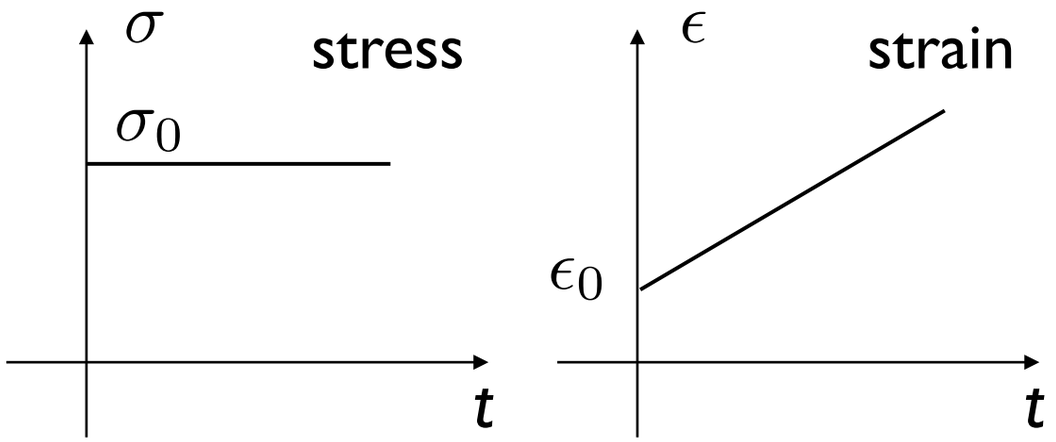
$$\sigma(t) = \sigma_0 \, \mathrm{e}^{-t/\tau_m}$$

where σ_0 is stress at time *t=0* and

$$\tau_m = \frac{\eta}{\mu}$$

is “Maxwell relaxation time”.

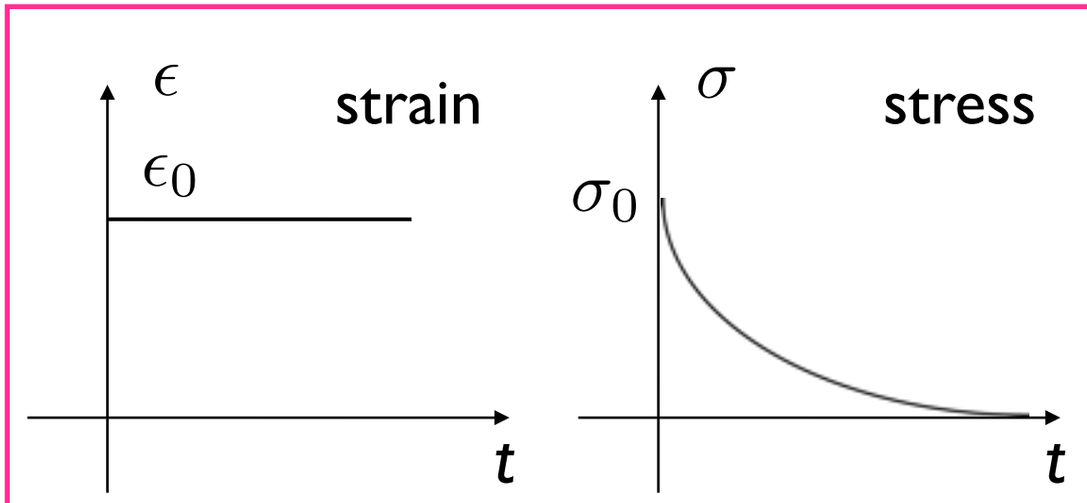
Thus, under a constant strain, Maxwell body “relaxes”.



CREEP

INPUT: $\sigma(t) = \sigma_0, \quad t \geq 0$

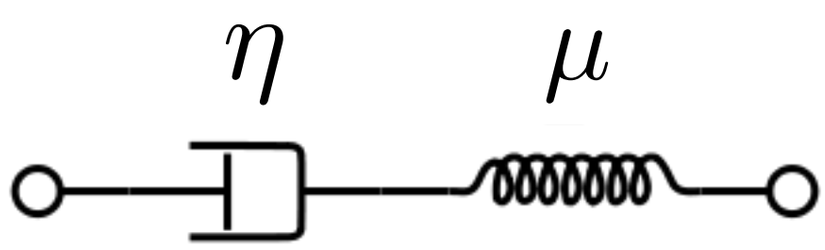
OUTPUT: $\epsilon(t) = \epsilon_0 + \dfrac{\sigma_0}{2\eta}\, t$



RELAXATION

INPUT: $\epsilon(t) = \epsilon_0, \quad t \geq 0$

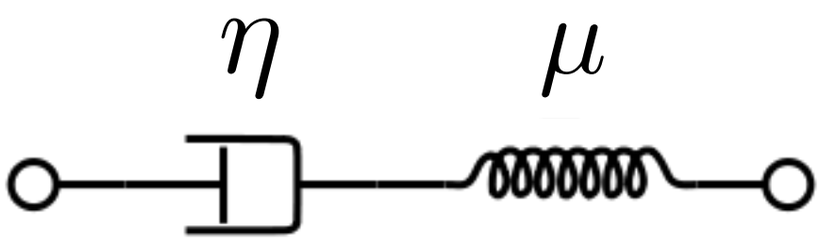
OUTPUT: $\sigma(t) = \sigma_0\ \mathrm{e}^{-t/\tau_m}$

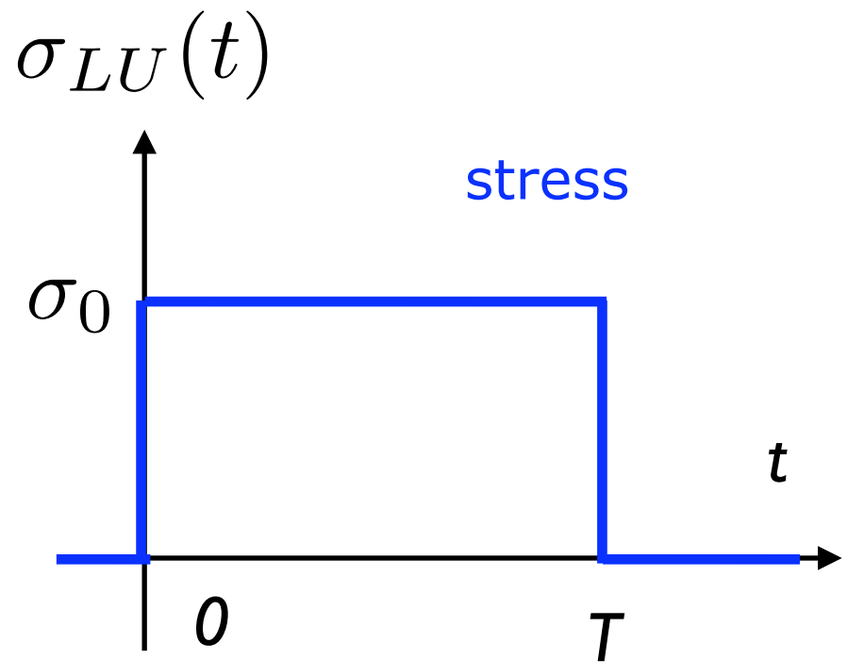


Maxwell body

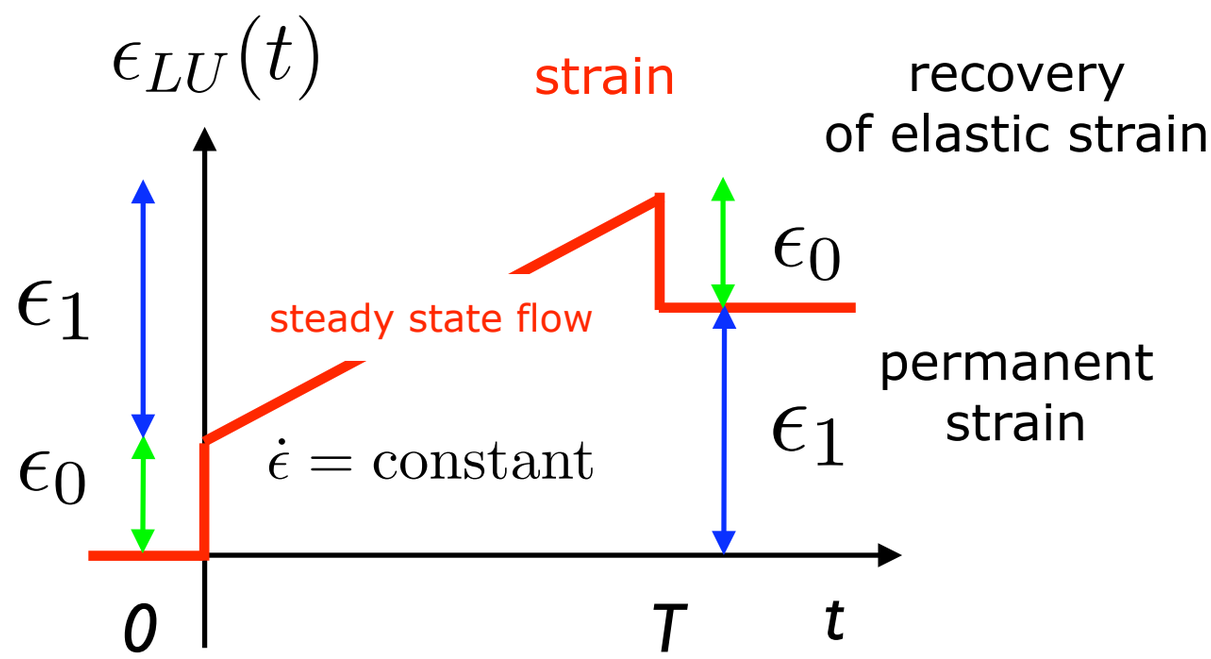
LOADING / UNLOADING experiment

$$\sigma_{LU}(t) = \sigma_0[H(t) - H(t-T)]$$





$$\epsilon_{LU}(t) = \frac{\sigma_0}{2\mu}[H(t) - H(t-T)] +$$
$$\frac{\sigma_0}{2\eta}(t - (t-T)H(t-\tau))$$



Recovery of elastic strain and permanent strain (liquid behavior).

Steady state flow observed until time t=T.

Deborah number

The Deborah number is a dimensionless number, used in rheology to characterize how "fluid" a material is.

Formally, the Deborah number is the ratio of a relaxation time, characterizing the intrinsic fluidity of a material, and the characteristic time scale of an experiment (or a computer simulation) probing the response of the material.

The Deborah number is:

$$\mathrm{D}^{(*)} = \frac{\tau_r}{\tau_o}$$

τ_r is relaxation time

τ_o is timescale of experiment

The smaller the Deborah number, the more fluid the material appears.

(*) KAPH, alfabeto ebraico

(**) The (1-page) 1964 Reiner manuscript is here: http://rrc.engr.wisc.edu/deborah.html

EFFECTIVE VISCOSITY is:

$$\eta_{eff} = \frac{1}{2\dot{C}} \qquad C = \frac{\text{strain}}{\text{unit stress}}$$

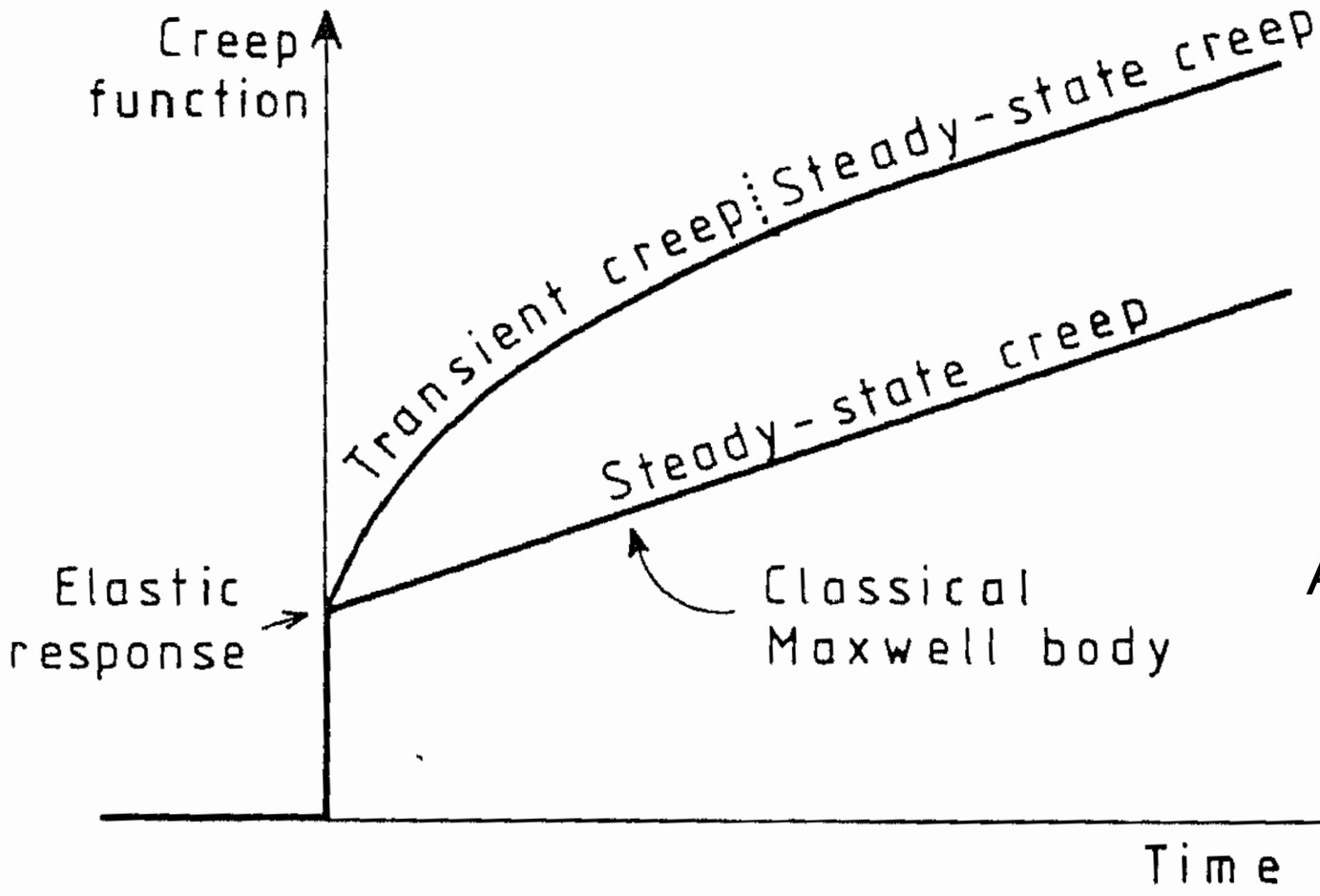
For the three rheological elements studied so far, effective viscosity is:

$$\eta_{eff} = \begin{cases} \infty & \text{H, HOOKE} \\ \eta & \text{N, NEWTON} \\ \eta & \text{M, MAXWELL} \end{cases}$$

A STEADY STATE rheology is characterized by $\eta_{eff} = \text{constant}$

In a TRANSIENT RHEOLOGY, η_{eff} is generally time-dependent!

Transient creep: a general feature of materials

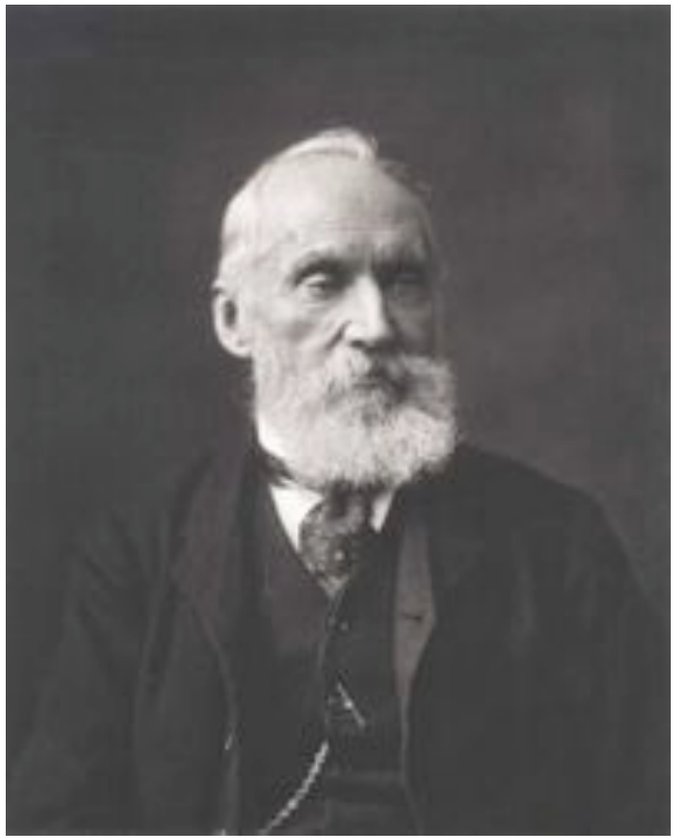


After *Muller (1986)*

Figure 1. Creep functions for the classical Maxwell body and for a material which exhibits transient creep in addition to steady-state creep. The long-time creep rates are assumed to be the same for both curves, i.e. the two bodies have the same steady-state viscosity.

There is "No viscosity", i.e., viscosity is time-dependent!

(even if the stress-strain relationship is linear...)

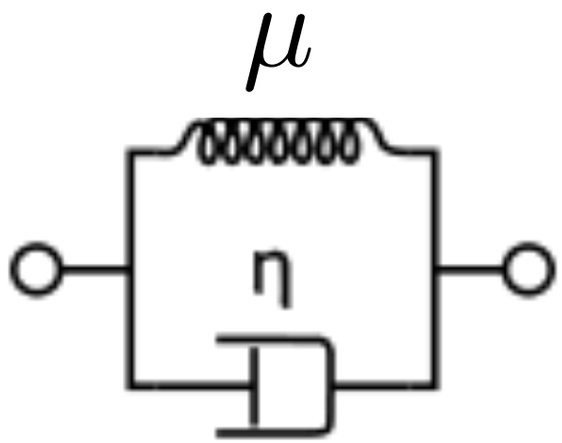


William Thomson
1st Baron of Kelvin
(1824 - 1907)



Woldemar Voigt
(1850 - 1919)

"Kelvin-Voigt body"



Kelvin-Voigt (KV)

KV = N | H

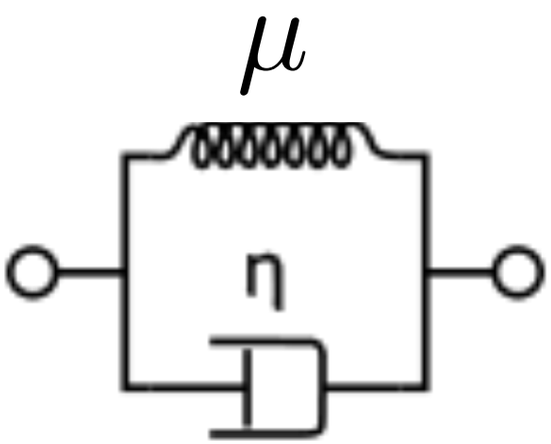
http://en.wikipedia.org/wiki/William_Thomson%2C_1st_Baron_Kelvin

http://en.wikipedia.org/wiki/Kelvin-Voigt_material

LOADING / UNLOADING experiment

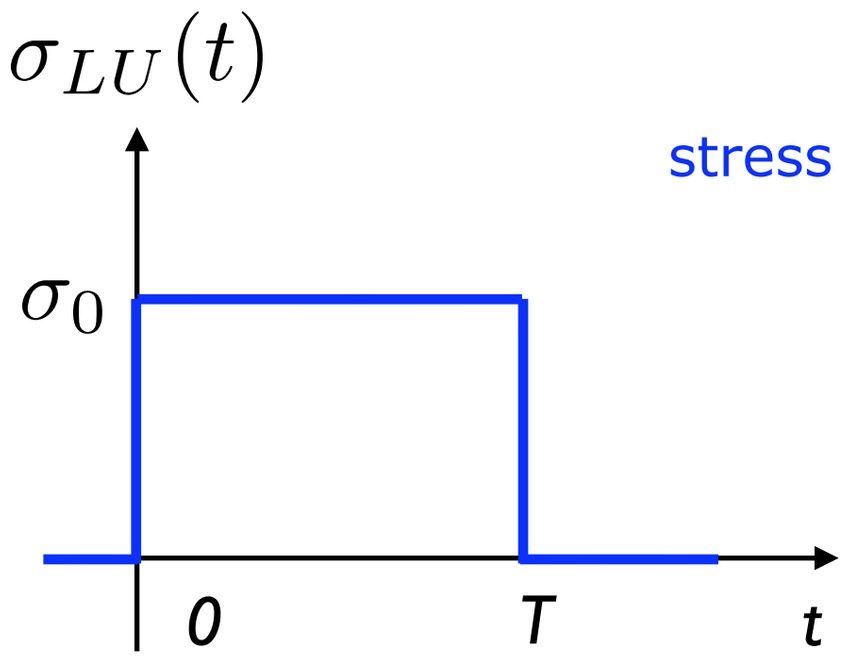
"Kelvin-Voigt body"

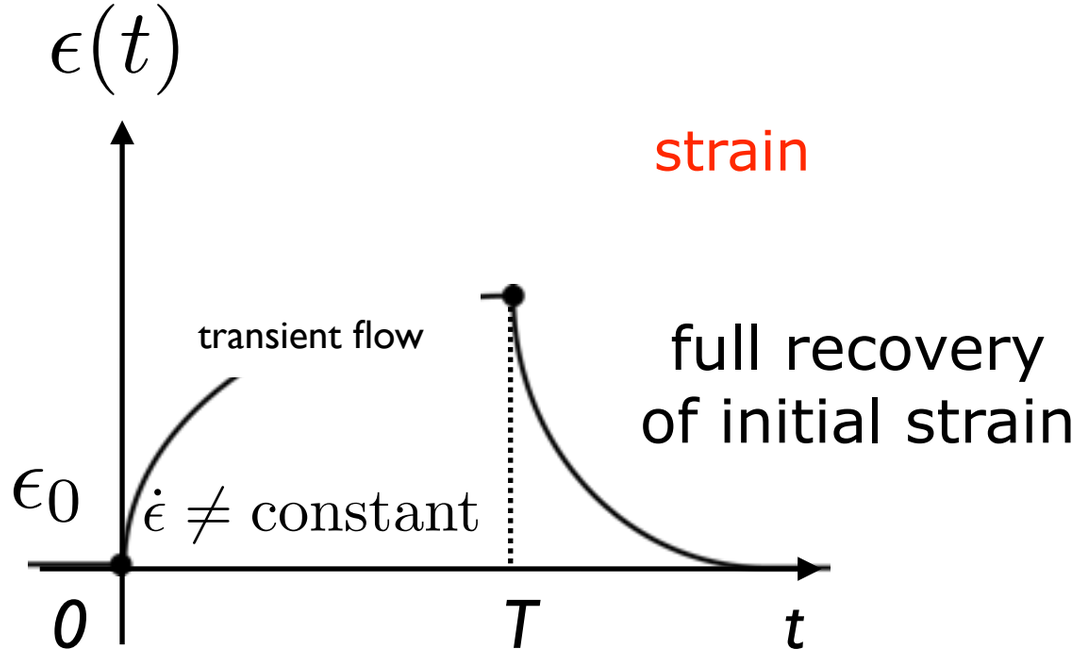
$$\sigma_{LU}(t) = \sigma_0[H(t) - H(t-T)]$$



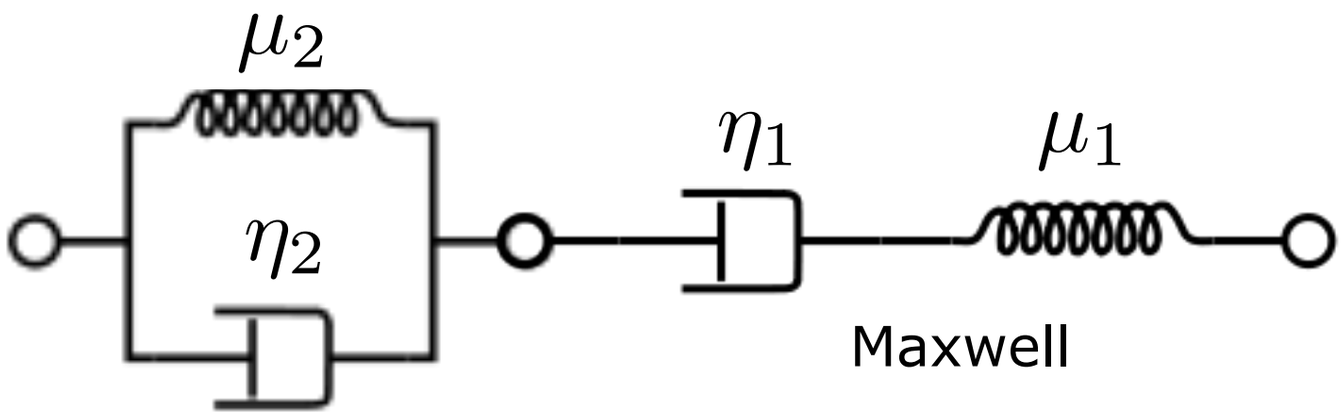
Kelvin-Voigt (KV)

$$\epsilon_{LU}(t) = \frac{\sigma_0}{2\mu}\left[\left(1 - e^{-\frac{\mu}{\eta}t}\right) - \left(1 - e^{-\frac{\mu(t-T)}{\eta}t}\right)H(t-T)\right]$$





"Delayed elasticity"

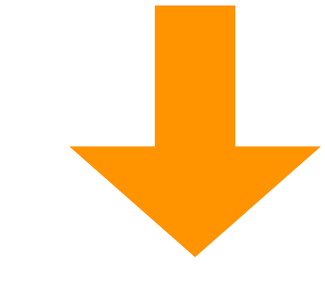




William Thomson
1st Baron of Kelvin
(1824 - 1907)



Woldemar Voigt
(1850 - 1919)





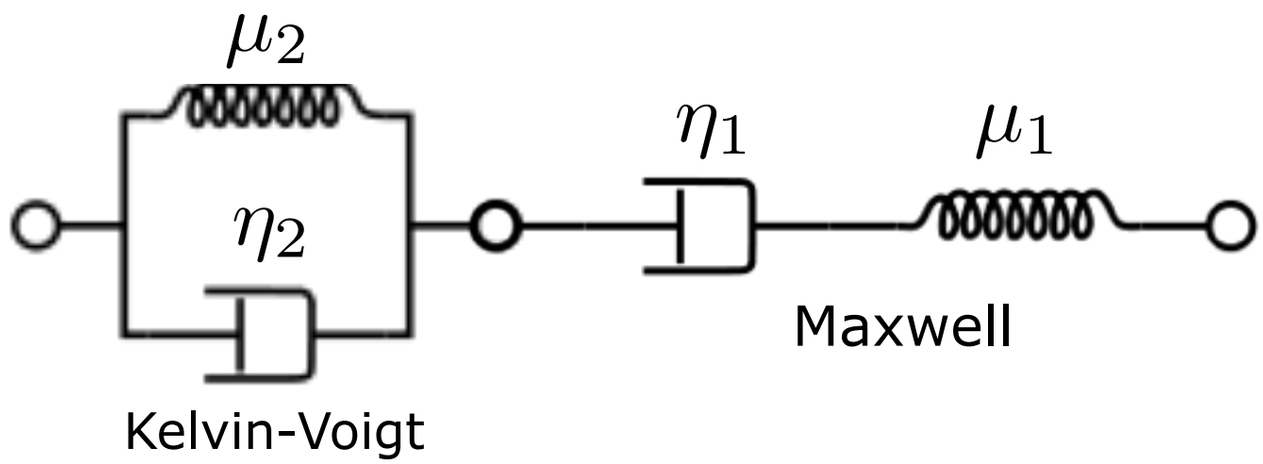
BURGERS, Johannes Martinus (1895-1981)

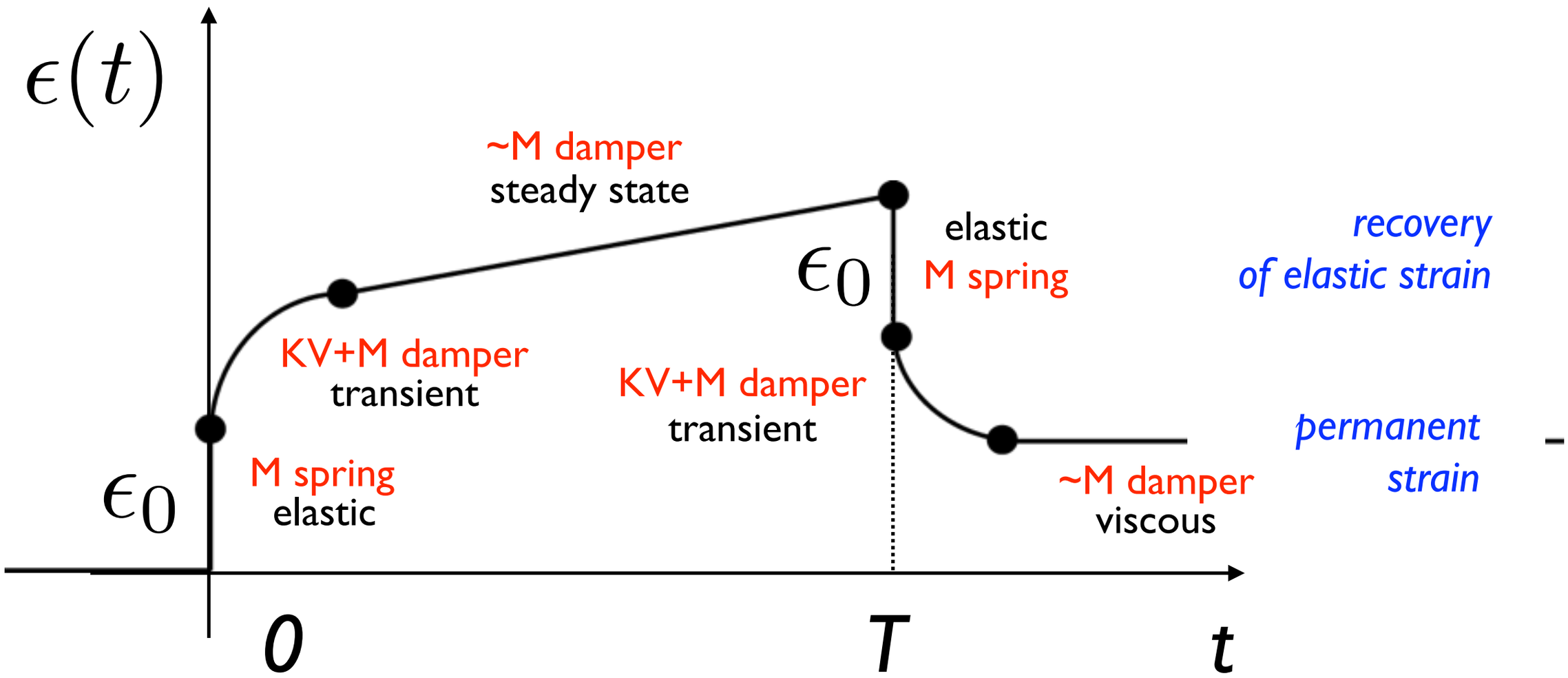
Burgers body (B)

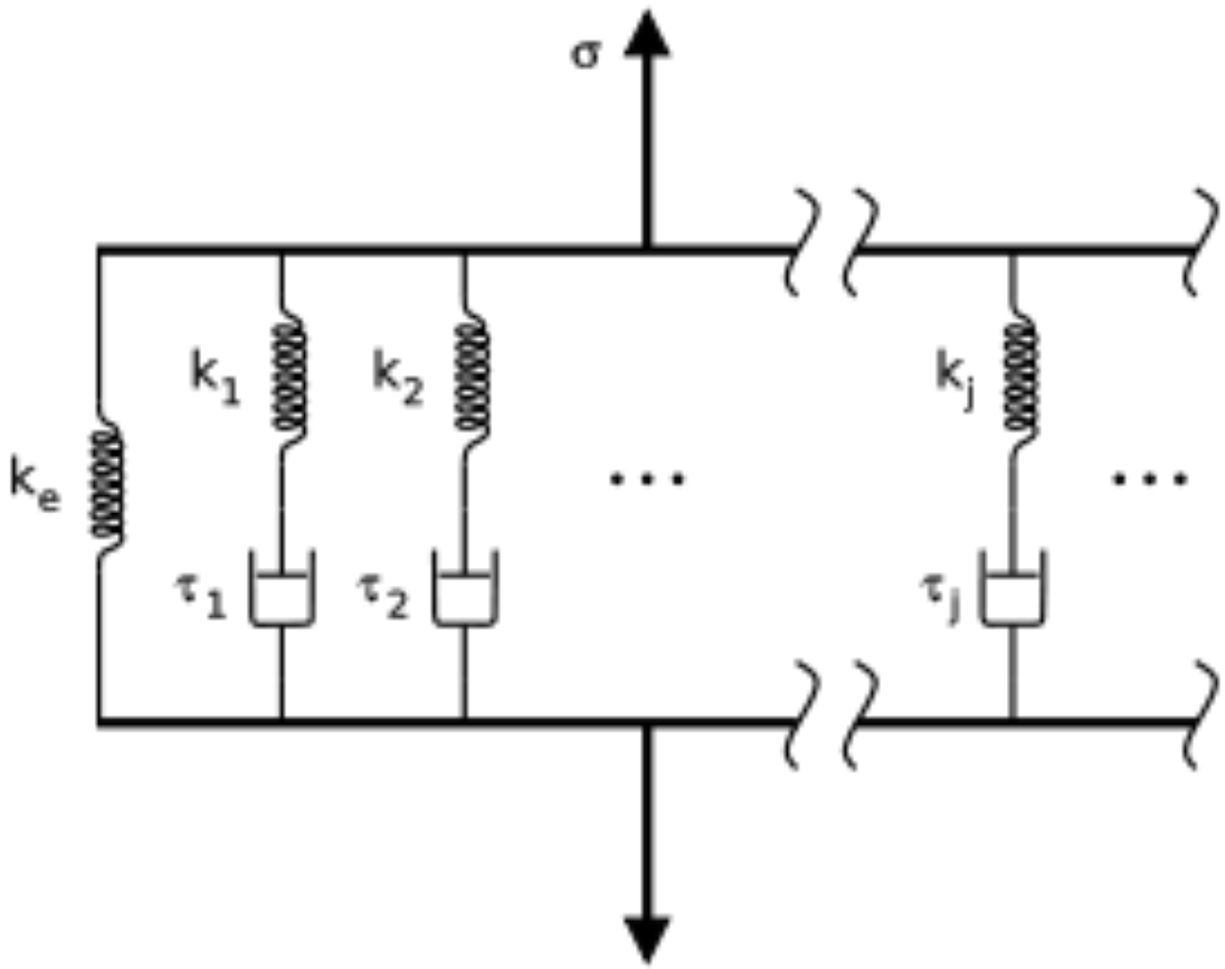
B = M - KV

A four-parameters rheological body

Burgers body







$MG = H \mid M_1 \mid \ldots \mid M_n$

n any integer OR n = infinity

The Generalized Maxwell **is the most general form** of the linear spring/dashpot model for viscoelasticity.

It takes into account that **the relaxation does not occur at a single time, but at a distribution of times**.

8). Some remarks...

Do all these rheologies have relevant applications ?

8). Some remarks...

Do all these rheologies have relevant applications ?

Yes!

8). Some remarks...

Do all these rheologies have relevant applications ?

Yes!

for example, to:

Amorphous and semicrystalline polymers,
Metals at very high temperature,
Mozzarella cheese,
Fruits, rice and potatoes,
Asphalt mixtures,
... ...

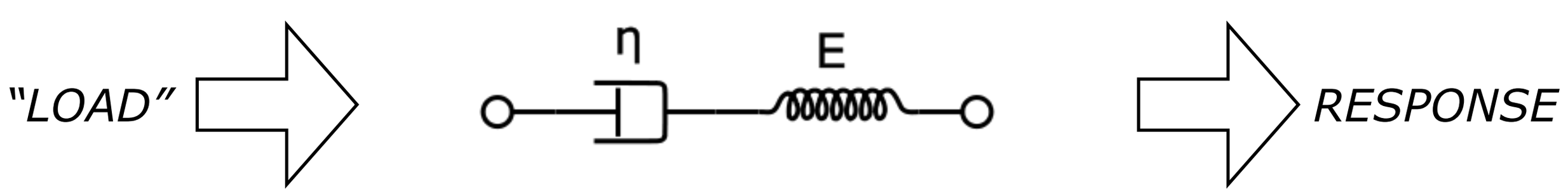
what about the **EARTH** ?

Which rheology for the Earth's mantle ?



PGR is a *natural creep experiment!*

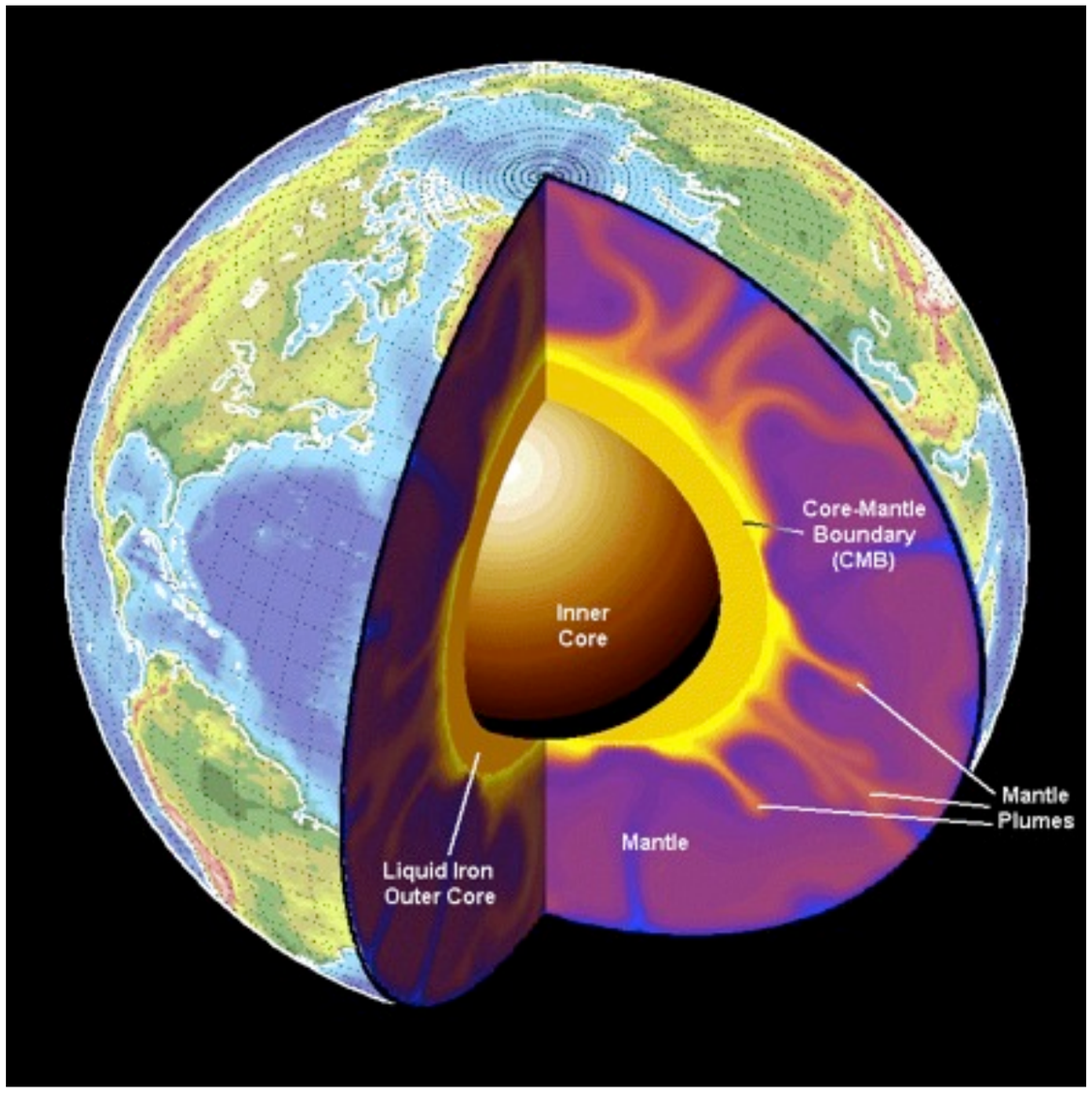
Which rheology for the Earth's mantle ?



PGR is a *natural creep experiment!*

Which rheology for the Earth's mantle ?

"LOAD"



RESPONSE

PGR is a *natural creep experiment!*

Guessing mantle viscosity

1. The shape of the uplift curve allows one to estimate the decay time,

2. Observed gravity anomalies indicates the amount of residual uplift,

3. This gives the order-of-magnitude of (Newtonian) mantle viscosity (Vening Meisnez, Haskell, 1935):

$$\eta \approx 10^{21} \mathrm{Pa\ s}$$

The Rocky terrain in shallow sea area

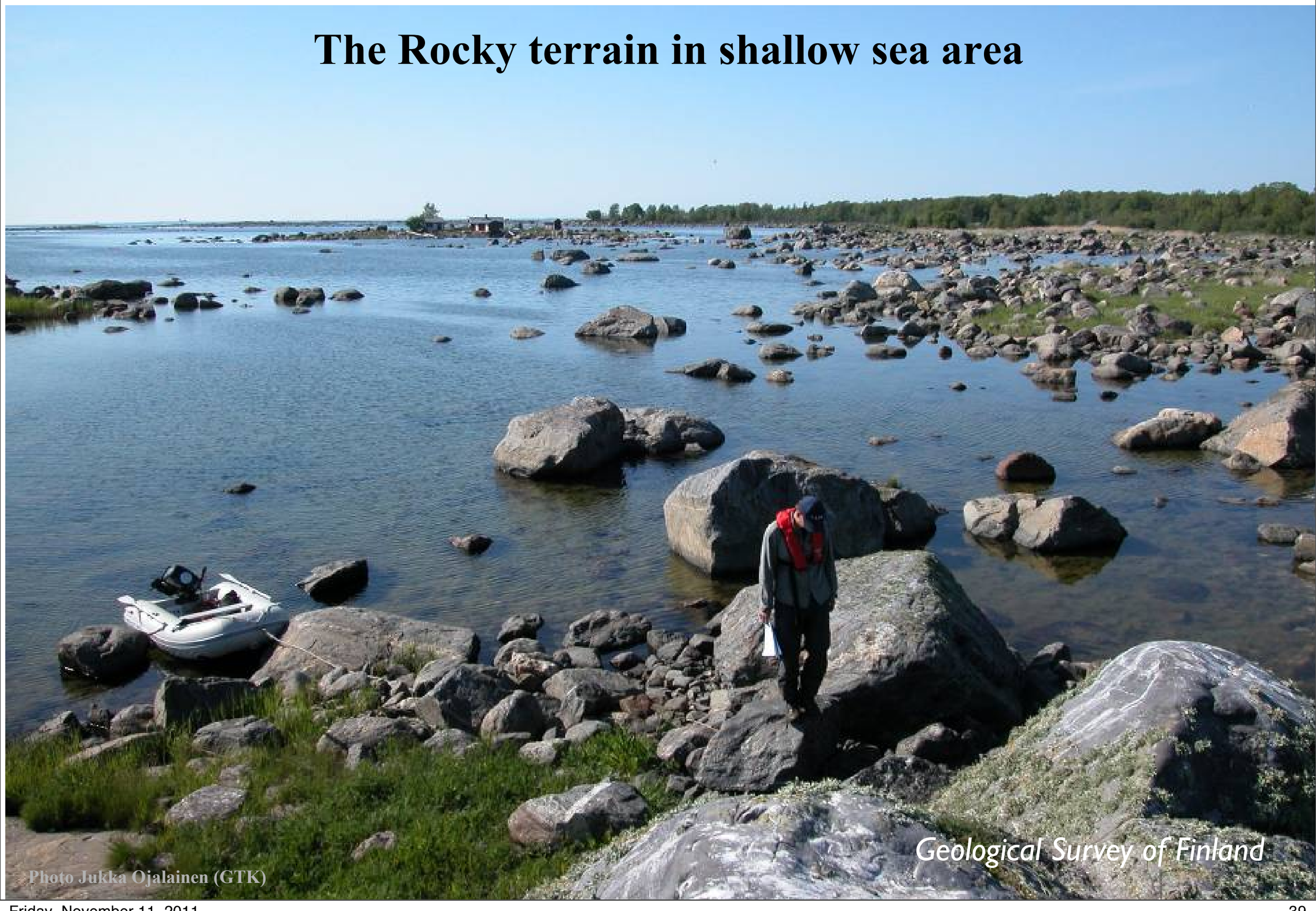


Photo Jukka Ojalainen (GTK)

Geological Survey of Finland

Outline

1). Basic things about Rheology

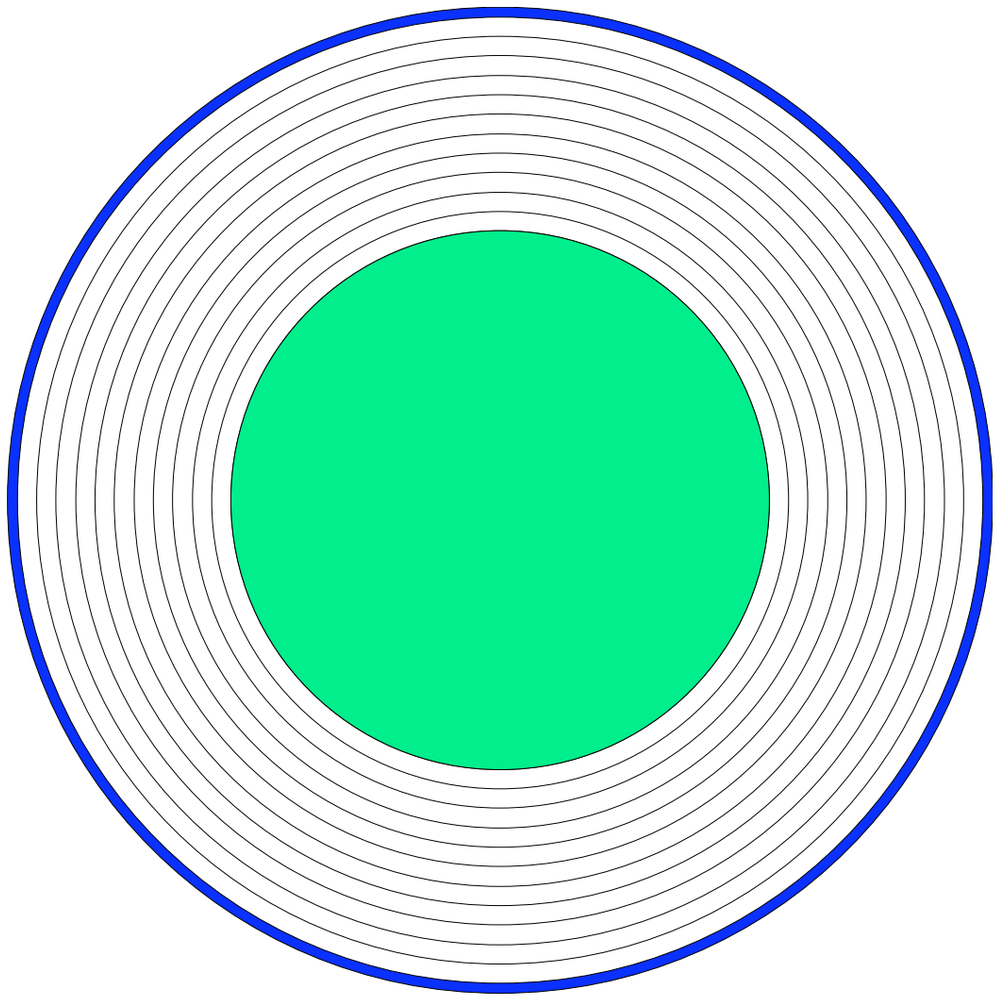
2). “Normal modes” (Spectra, Love numbers...)

3). “Loading problems” (recent PGR benchmark activity)

4). Special topic: non-traditional computation of Love numbers

Charles University in Prague, Department of Geophysics
November 7–11, 2011

Green functions for layered Earth models



✓Spherically symmetric Earth:

$$\mu = \mu(r)$$
$$\lambda = \lambda(r)$$

Lame constants

$$\rho = \rho(r)$$

Density

$$\eta = \eta(r)$$

Viscosity

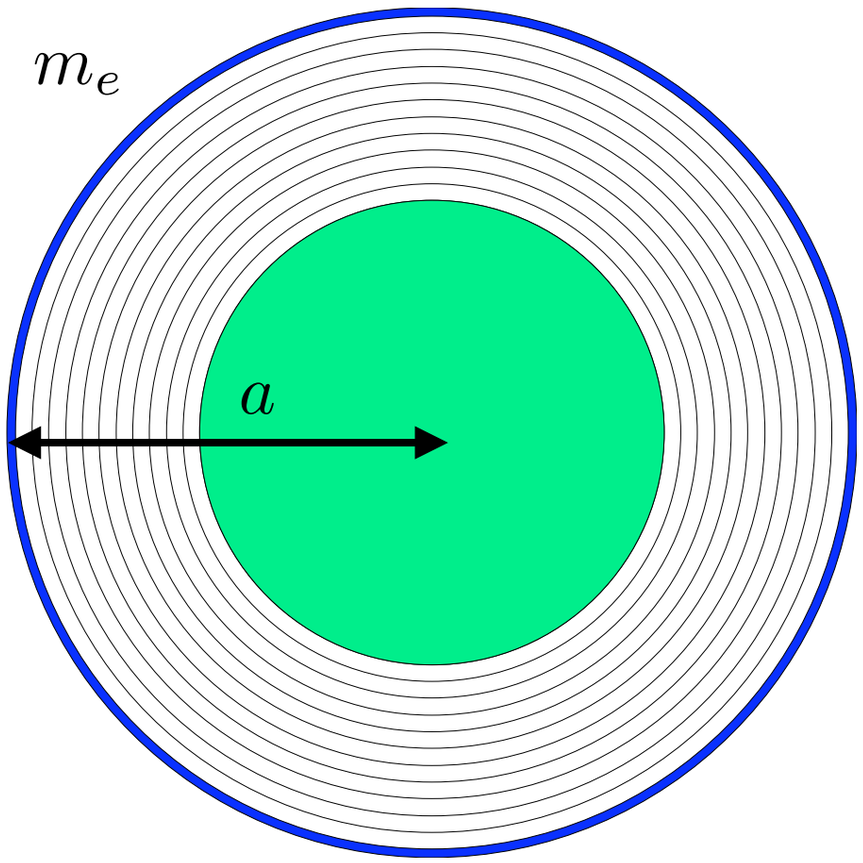
✓Incompressibility: $\lambda = \infty$

✓Inviscid layers: $\eta = 0$

✓Effective elasticity: $\eta = \infty$

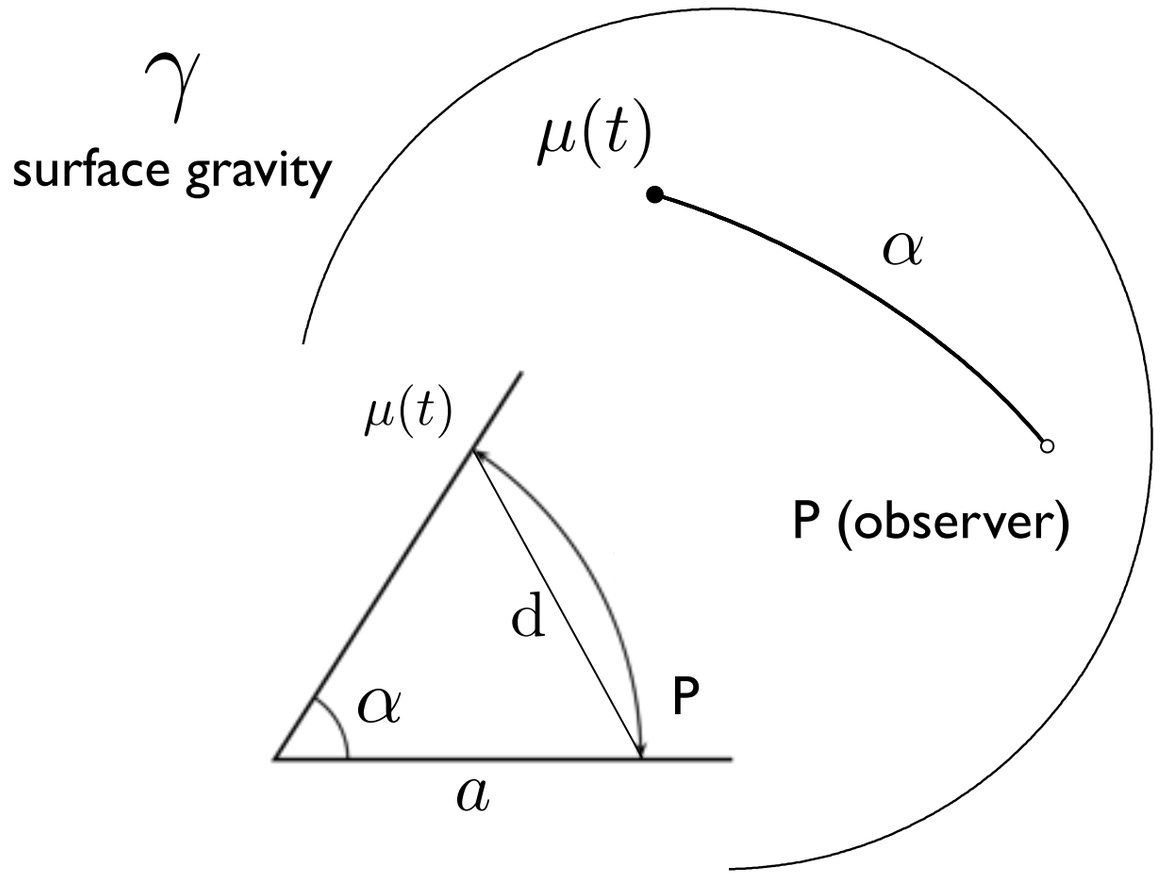


Spherically symmetric Earth



Spherically symmetric Earth

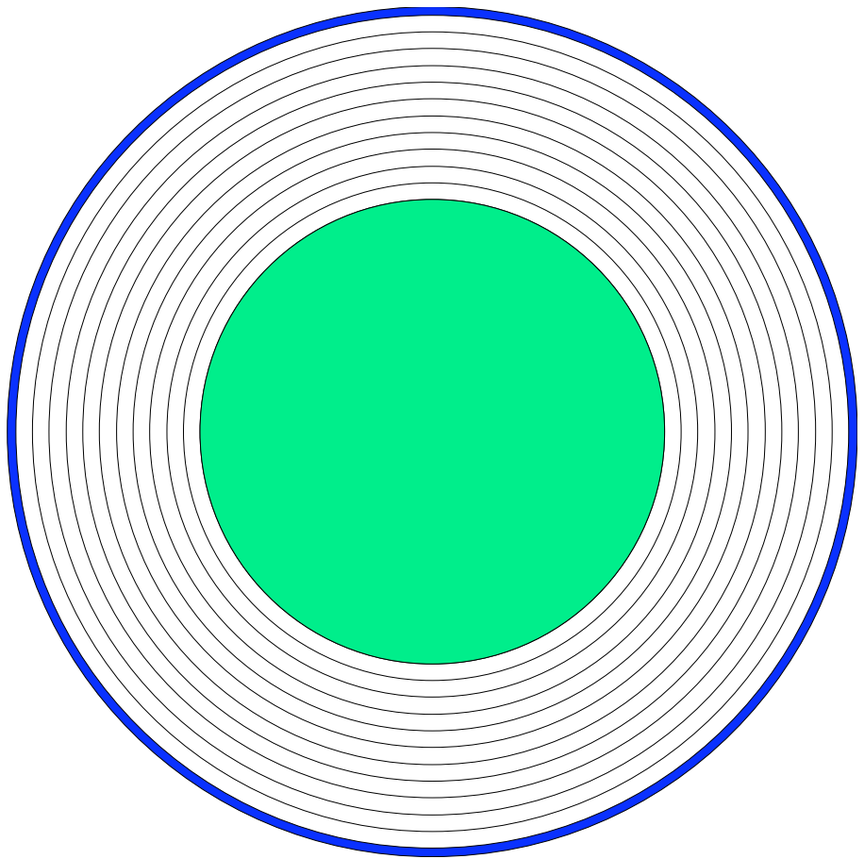
Source-observer geometry



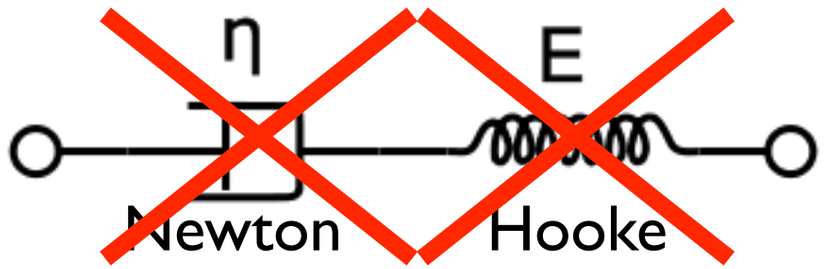
Source-observer geometry

LOCALIZED and IMPULSIVE source: $\mu(t) = \delta(t) m_s$





RIGID Earth


η
E
Newton
Hooke


COST ACTION ES0701
improved constraints on models of GIA

$$\phi^r(\alpha,t) = \frac{G\mu(t)}{d(\alpha)}$$

(Variation of the) gravity potential at P, by **Newton law**.

$$G^r_\phi(\alpha,t) \equiv \frac{\phi^r(\alpha,t)}{m_s}$$

GREEN FUNCTION (GF) for gravity potential. AKA "Direct GF"

By simple geometry and the definition of Legendre polynomials, we obtain the **harmonic decomposition**:

$$G^r_\phi(\alpha,t) = \delta(t)\sum_{l=0}^{\infty}\phi^r_l P_l(\cos\alpha) \qquad \phi^r_l \equiv \frac{a\gamma}{m_e}$$

For a RIGID EARTH, the GF is spherically symmetric wrt the source and "in phase" with it.

The first few Legendre polynomials

$x = \cos\theta$

$$P_0(x) = 1 \qquad \textit{degree 0}$$

$$P_1(x) = x \qquad \textit{1}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad \textit{2}$$

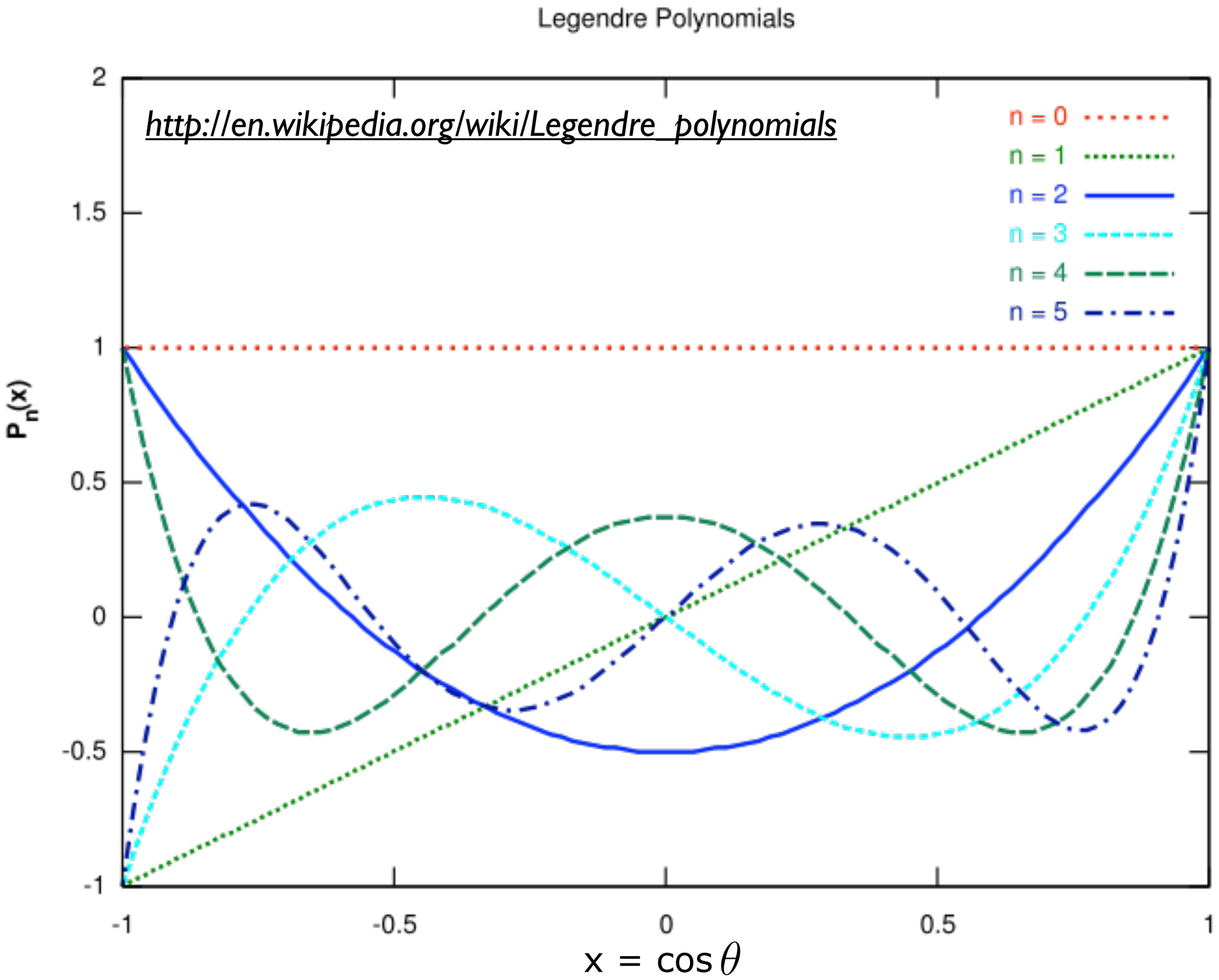
$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \qquad \textit{3}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad \textit{4}$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \qquad \textit{5}$$

… … … … … …

$P_n(x)$ = a polynomial of degree "n" *n*


Legendre Polynomials
http://en.wikipedia.org/wiki/Legendre_polynomials
n = 0
n = 1
n = 2
n = 3
n = 4
n = 5
$P_n(x)$
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
$x = \cos\theta$

"Laplace equation"

$$\nabla^2 f = 0$$

Hamonic functions "f" are solutions of Laplace equation

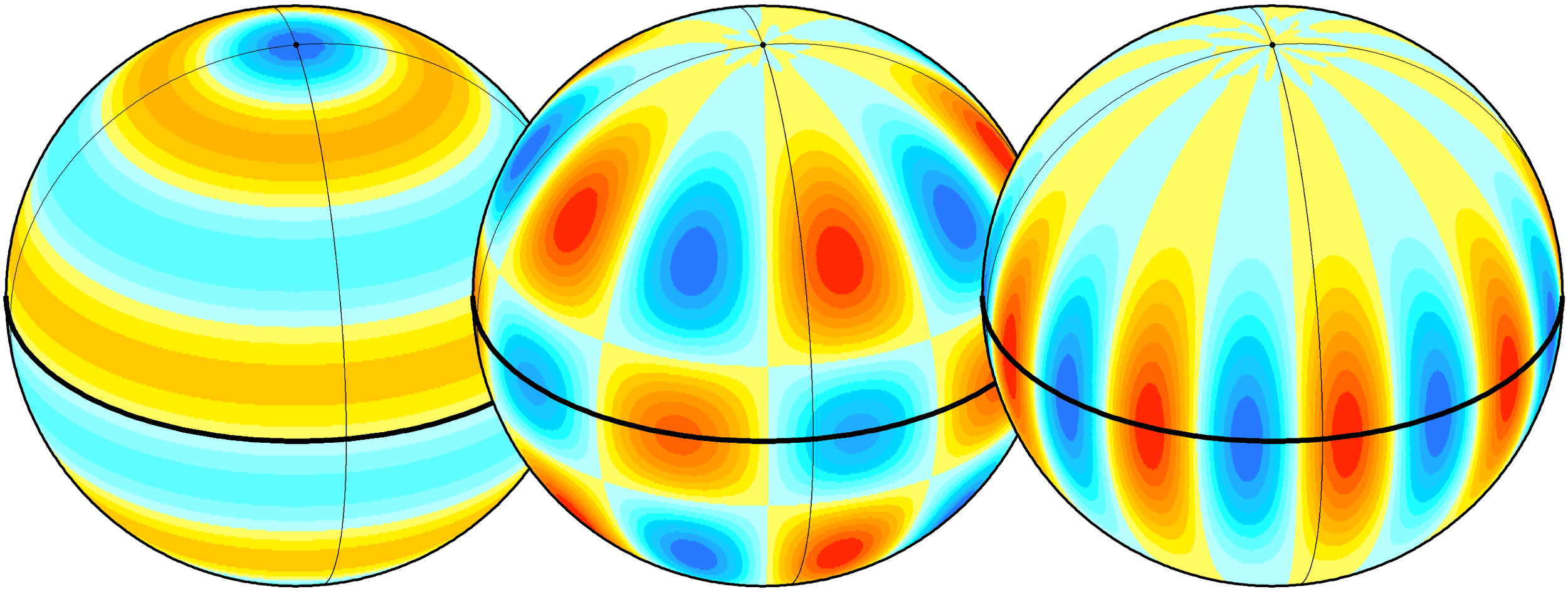
Real spherical harmonics

Zonal harmonics

(7, 0)

(7,5)

(9,9)



Zonal

Tesseral

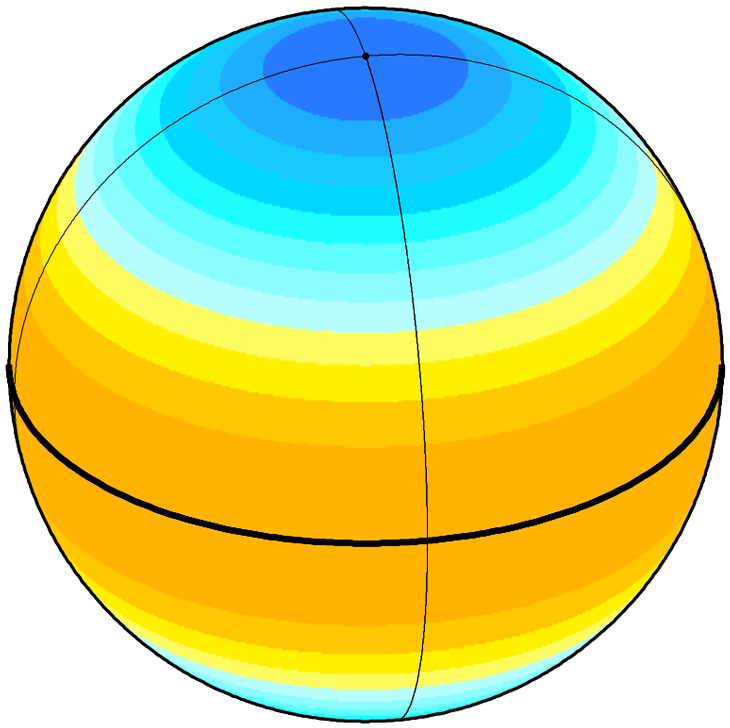
Sectorial



Real spherical harmonics

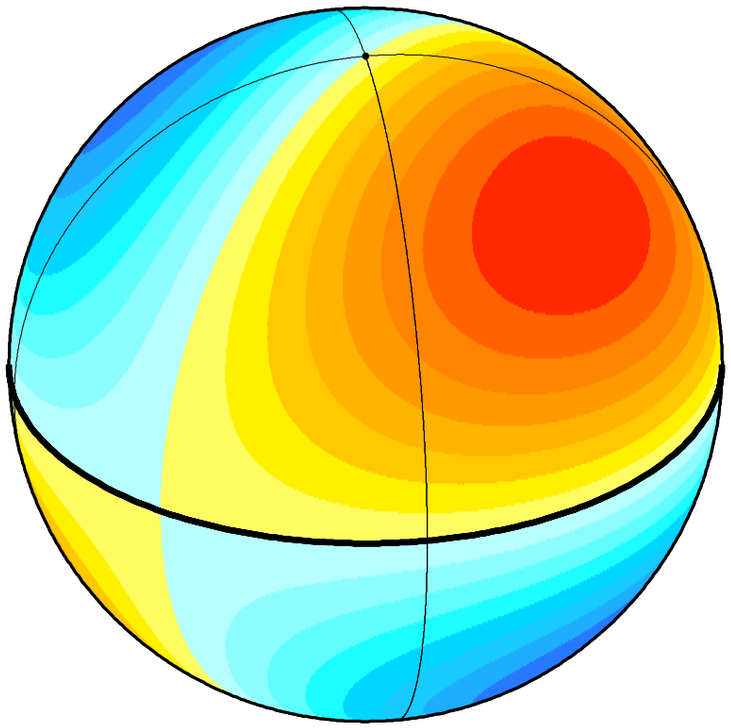
Degree two harmonics

(2, 0)



$$S_{20} = A(3\cos^2\theta - 1)$$

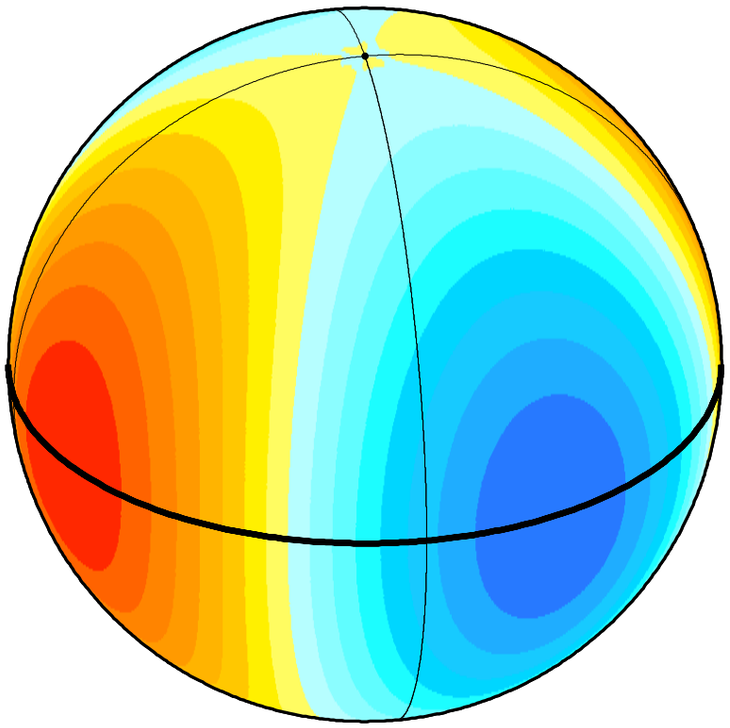
(2, 1)



$$S_{21} = B\sin\theta\cos\theta\cos\lambda$$

$$S_{21} = C\sin\theta\cos\theta\sin\lambda$$

(2, 2)

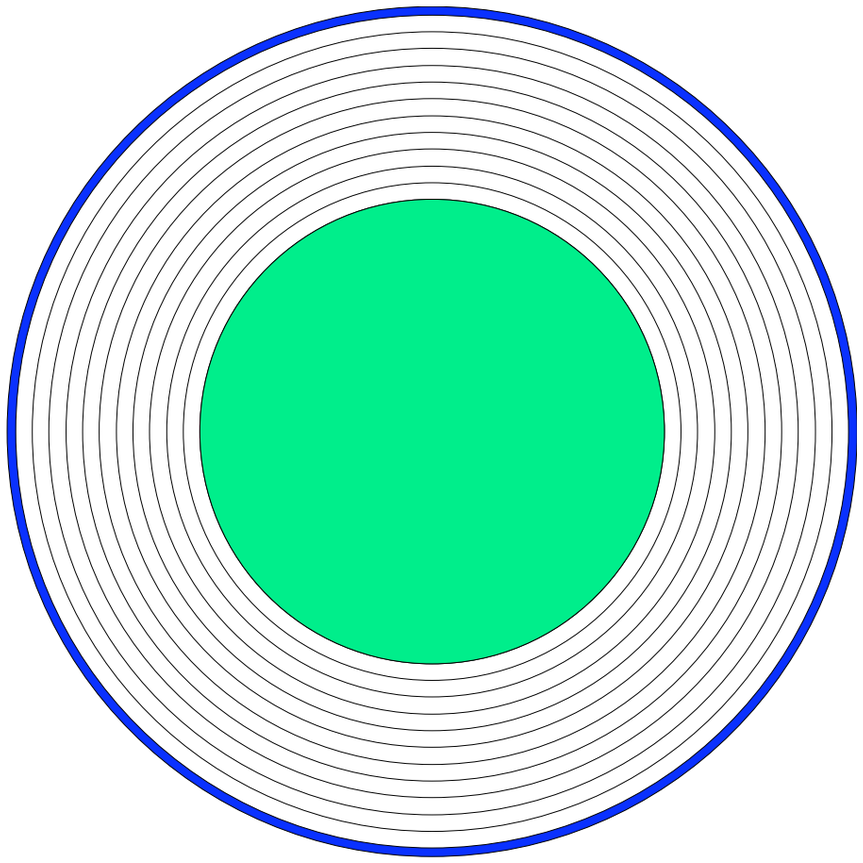


$$S_{22} = D\sin^2\theta\cos 2\lambda$$

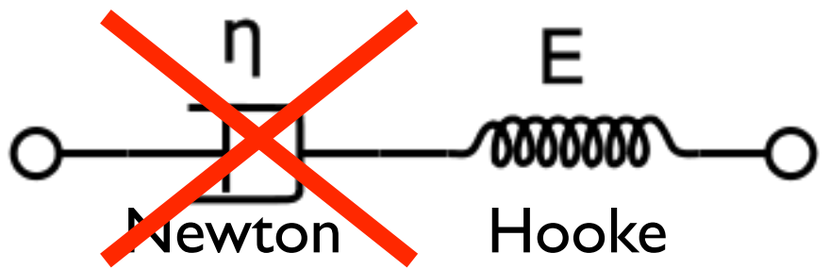
$$S_{22} = E\sin^2\theta\sin 2\lambda$$

They play a role in the physics of Earth rotation

TIDAL POTENTIAL is a degree two function



ELASTIC Earth



For an elastic Earth we observe two new effects:

1) Since the Earth changes its shape, its gravity field will be subject to a further modification that adds to that observed for a rigid Earth.

Note:
since the Earth is elastic (remember, *"sic tensio ut vis"*, this INCREMENTAL potential will be in phase with the source)

2) The Earth yields UNDER THE LOAD, and a displacement field arises as a consequence of yielding.

Note:
since the Earth is spherically symmetric, the displacement field is a vector with only two components: one UP, the other along the source-observer meridian

Now we look separately to these two effects.

1) For the INCREMENTAL potential we write:

$$G^e_\phi(\alpha, t) = \delta(t) \sum_{l=0}^{\infty} \phi^e_l P_l(\cos\alpha).$$

where, the coefficients will be proportional *(since tensio...)*, harmonic degree-by-degree, to the "rigid" coefficients:

$$\phi^e_l = k^e_l \phi^r_l$$

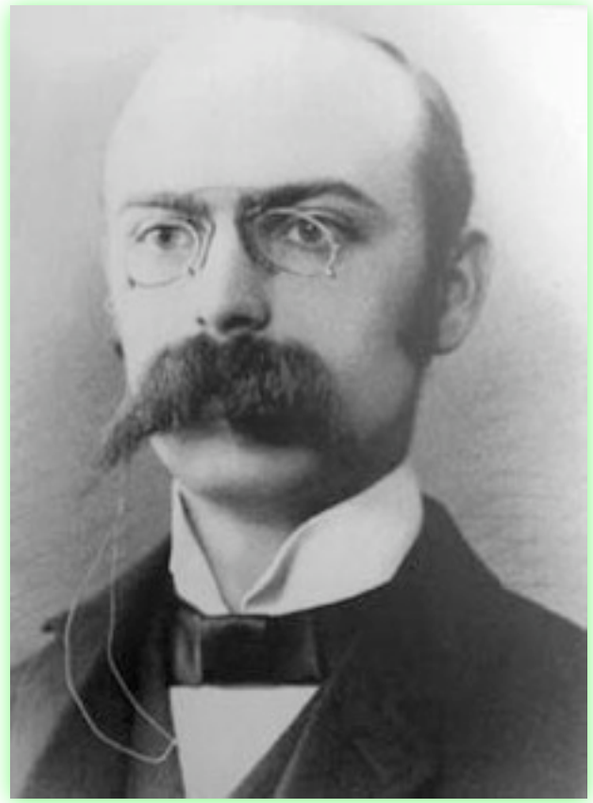
and where the dimensionless number:

$$k^e_l$$

is called "*Load-dedormation coefficient*" (LDC) of degree "*l*" for the incremental potential relative to an elastic spherically symmetric Earth.

A few remarks:

Augustus Edward Hough Love (17 April 1863 in Weston-super-Mare - 5 June 1940 in Oxford)



k_l^e depends, in general, on

✓the geometry of the spherical model,

✓the thickness of layers,

✓the numerical values of density and elastic constants.

Sometimes, the LDCs are referred to as “Love numbers”, after the great English matematician Love (*), even if this should be done only for TIDAL LOADING (here we discuss SURFACE LOADING).

Computing k_l^e (*any* LDC) demands the solution of the Navier-Cauchy EQUILIBRIUM EQUATIONS for the chosen Earth model.

The TOTAL elastic Green Function for the gravity potential stems from the DIRECT ("rigid") and INDIRECT terms:

$$G^r_\phi(\alpha,t) = \delta(t)\sum_{l=0}^{\infty}\phi^r_l P_l(\cos\alpha) \qquad \text{"DIRECT GF"}$$

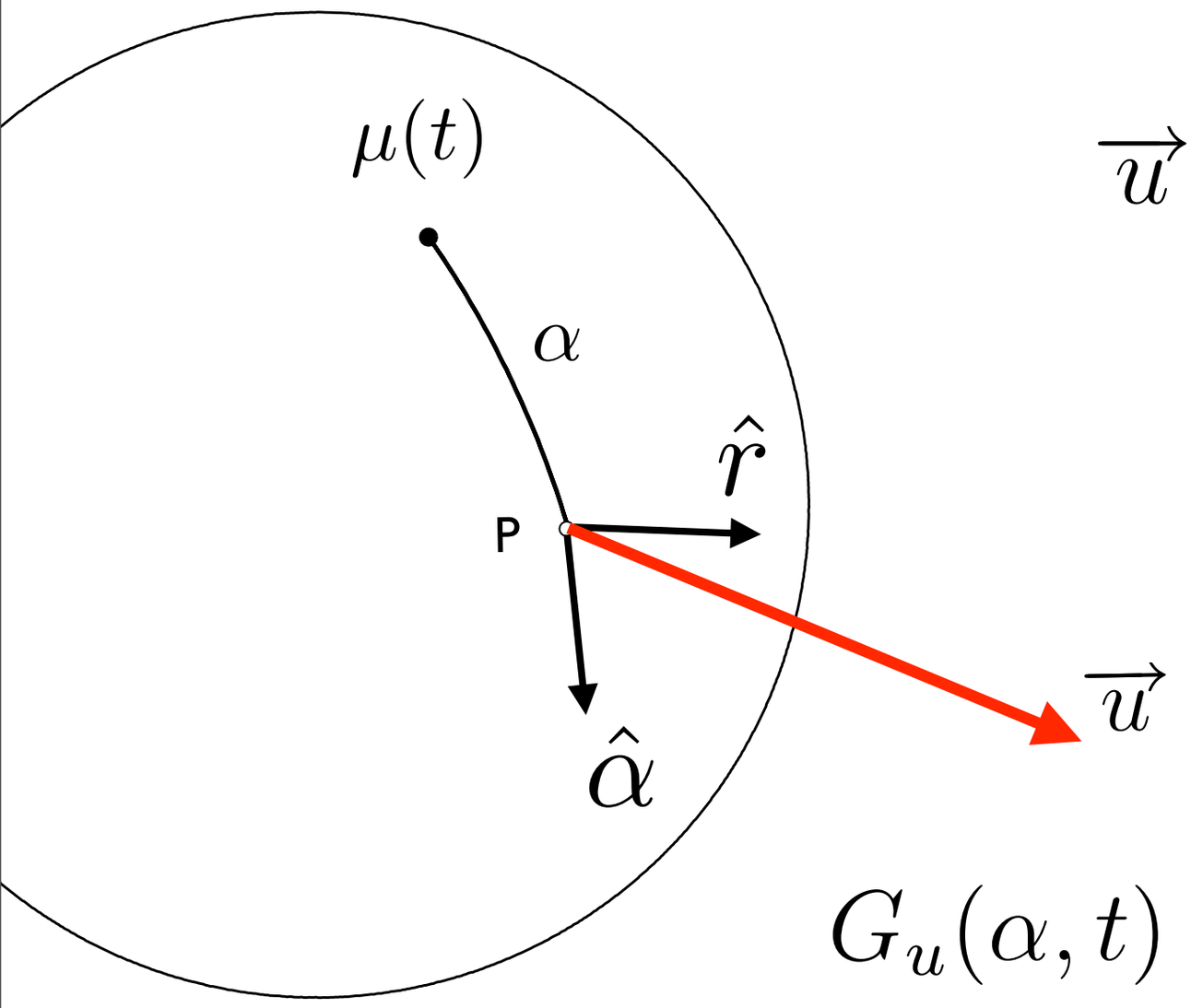
$$G^e_\phi(\alpha,t) = \delta(t)\sum_{l=0}^{\infty}\phi^e_l P_l(\cos\alpha). \qquad \text{"INCREMENTAL GF"}$$

With $\phi^e_l = k^e_l \phi^r_l$ and $\phi^r_l \equiv \dfrac{a\gamma}{m_e}$

the **TOTAL** GF is:

$$G_\phi(\alpha,t) \equiv G^r_\phi + G^e_\phi = \delta(t)\frac{a\gamma}{m_e}\sum_{l=0}^{\infty}(1+k^e_l)P_l(\cos\alpha)$$

2) The displacement field is due to elastic yielding:



Displacement:

$$\vec{u} = G_u(\alpha, t)\hat{r} + G_v(\alpha, t)\hat{\alpha}$$

Green function for **vertical** displacement:

$$G_u(\alpha, t)$$

Harmonic decomposition:

$$G_u(\alpha, t) = \delta(t)\frac{a}{m_e}\sum_{l=0}^{\infty} h_l^e P_l(\cos\alpha)$$

h_l^e is the Love number for vertical displacement.

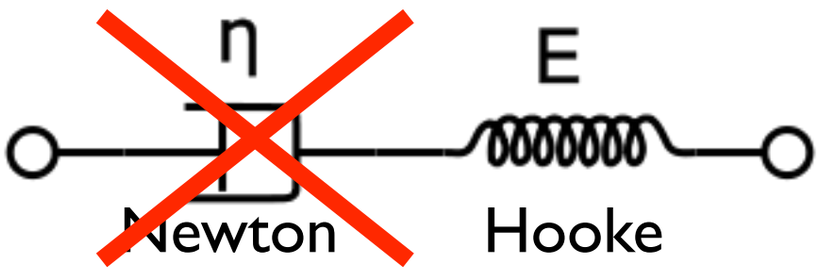
So, we summarize the GFs for an elastic Earth as:

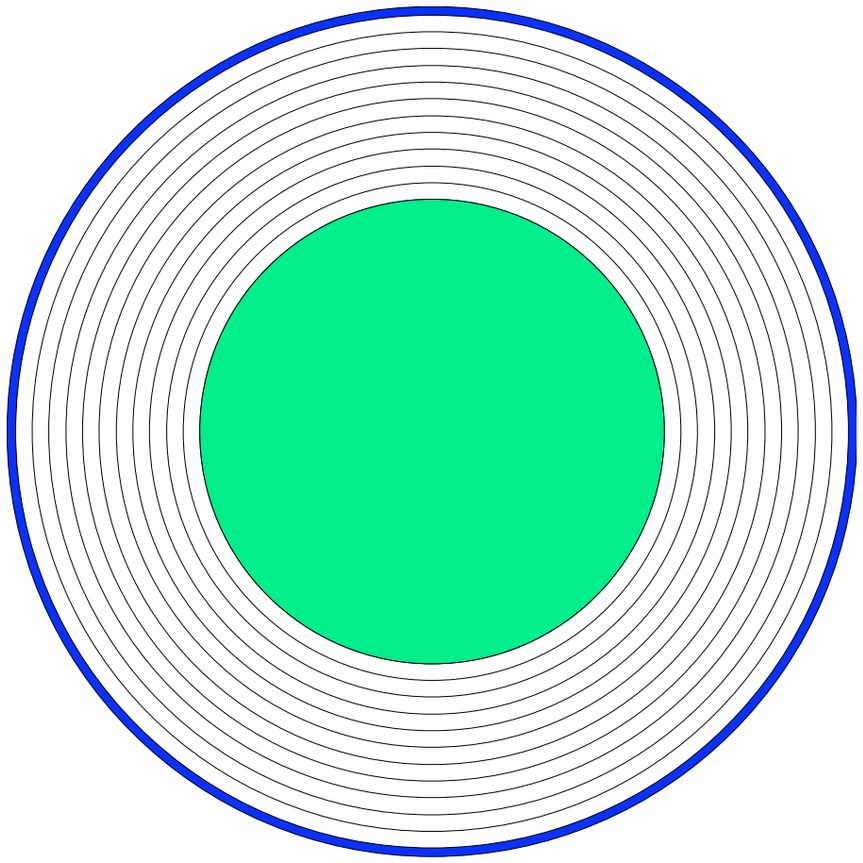
$$\left\{ \begin{array}{c} \frac{1}{\gamma} G_\phi \\ G_u \end{array} \right\} (\alpha, t) = \frac{a}{m_e} \sum_{l=0}^{\infty} \left\{ \begin{array}{c} k_l \\ h_l \end{array} \right\} (t) \; P_l(\cos\alpha)$$

where:

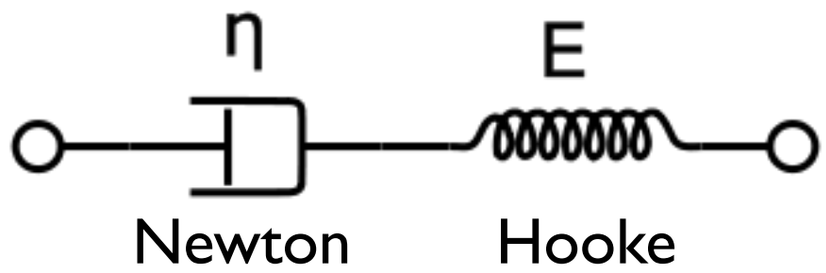
$$\left\{ \begin{array}{c} k_l \\ h_l \end{array} \right\} (t) = \left\{ \begin{array}{c} 1 + k_l^e \\ h_l^e \end{array} \right\} \delta(t)$$

are the 'time-dependent Love numbers' for an elastic Earth.

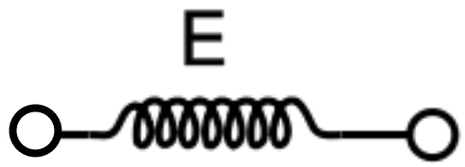




VISCO - ELASTIC Earth

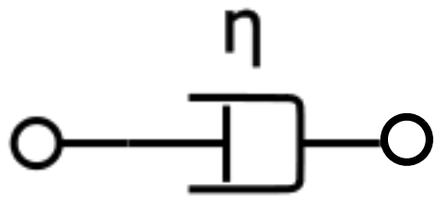


Seismology



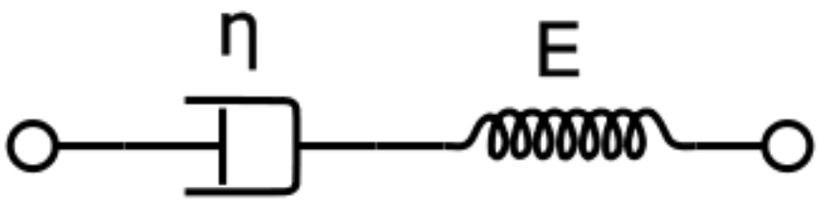
Hooke

Mantle convection

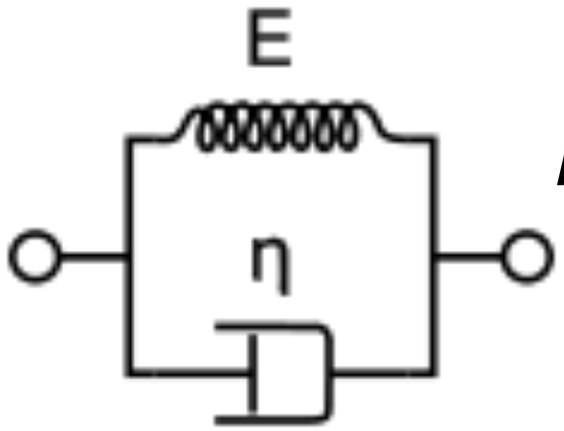


Newton

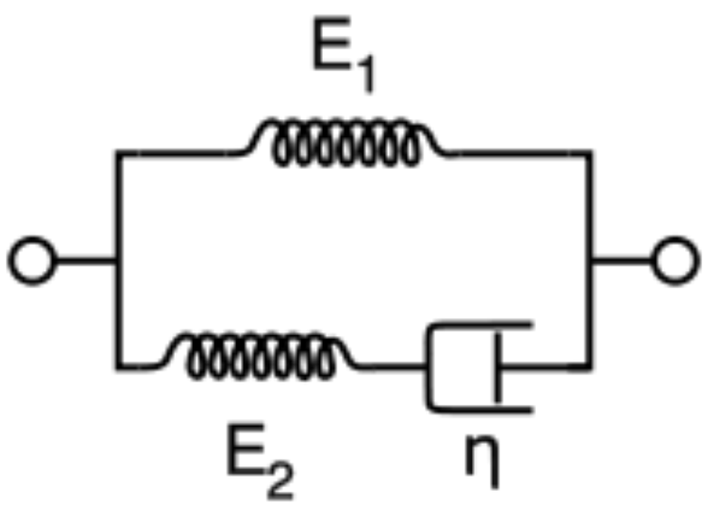
PGR



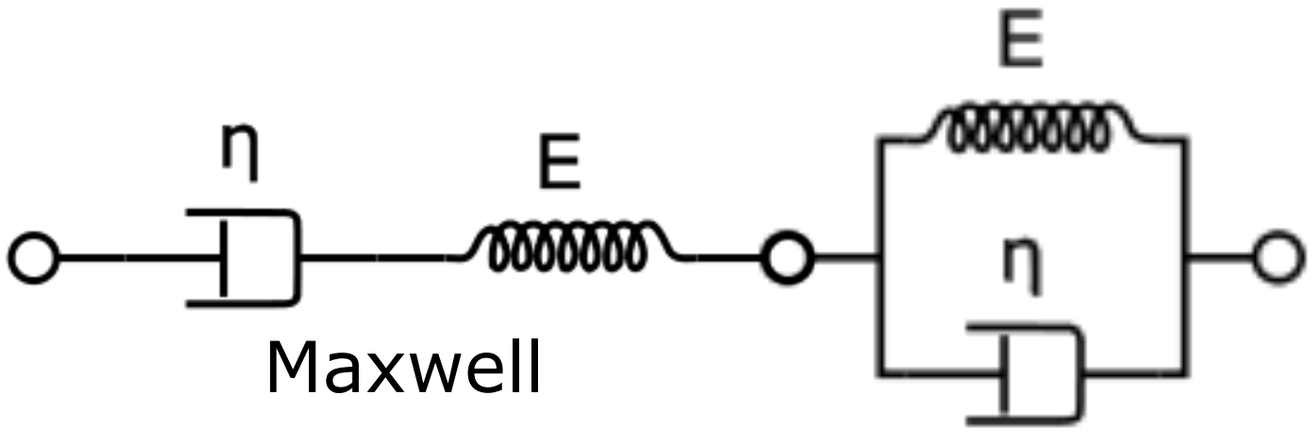
Maxwell



Kelvin-Voigt



Standard Linear Solid



Maxwell

Kelvin

Burgers

The response of the Earth is not instantaneous due to "imperfections of elasticity".

From the **literature**, we know that for a visco-elastic Earth, the LDCs are the sum of an *elastic* and a *viscous* component:

$$\left\{ \begin{array}{c} k_l \\ h_l \end{array} \right\}(t) = \left\{ \begin{array}{c} 1+k_l^e \\ h_l^e \end{array} \right\}\delta(t) + H(t)\sum_{j=1}^{M}\left\{ \begin{array}{c} k_{lj} \\ h_{lj} \end{array} \right\} e^{-t/\tau_{lj}}$$

with:

(h_l^e, k_l^e) *Elastic* Love numbers

(h_{lj}, k_{lj}) *Viscous* Love numbers

τ_{lj} Characteristic relaxation times

These quantities describe the "**viscoelastic normal modes**".

- Computing the LDCs demands solving the following system of equations, degree per degree (*):

Momentum equation	*Momentum is conserved*
Laplace equation	*The GP is harmonic outside*
Continuity equation	*Mass is conserved*
Constitutive equation	*Sic tensio... or Maxwell, or..*
Liouville equations	*Angular momentum is conserved*

Equations are usually solved in the Laplace transform domain...



Geophysical Journal International
Volume 11 Issue 1 Page 133 - September **1966**

Computation of Love Numbers and Load Deformation Coefficients for a Model Earth

I. M. Longman
Department of Applied Mathematics, Weizmann Institute of Science, Rehovot, Israel

Summary

Computations of Love numbers and load deformation coefficients for a Gutenberg model Earth are described.

The computations present some special features which are absent in the calculation of the periods of free oscillation of the Earth. This is due to the fact that the absence of inertia terms in the static problem involves a certain indeterminacy in the particle displacement in the liquid core. It is found that a necessary condition for the existence of a solution of the problem of the static deformation of the Earth under surface mass loads or externally applied gravity fields is that the Adams-Williamson condition holds in the core. This is without any assumption as to chemical homogeneity, but is merely a consequence of the assumption of zero **rigidity in the core**. Another special feature is the computation of the load deformation coefficient h'0(of order zero), which has a significance even though the ordinary Love number h0 does not exist. h'0 has not previously been evaluated by other authors, nor has its existence been noted. Three special dimensionless constants play a significant role in the calculations. ... ...

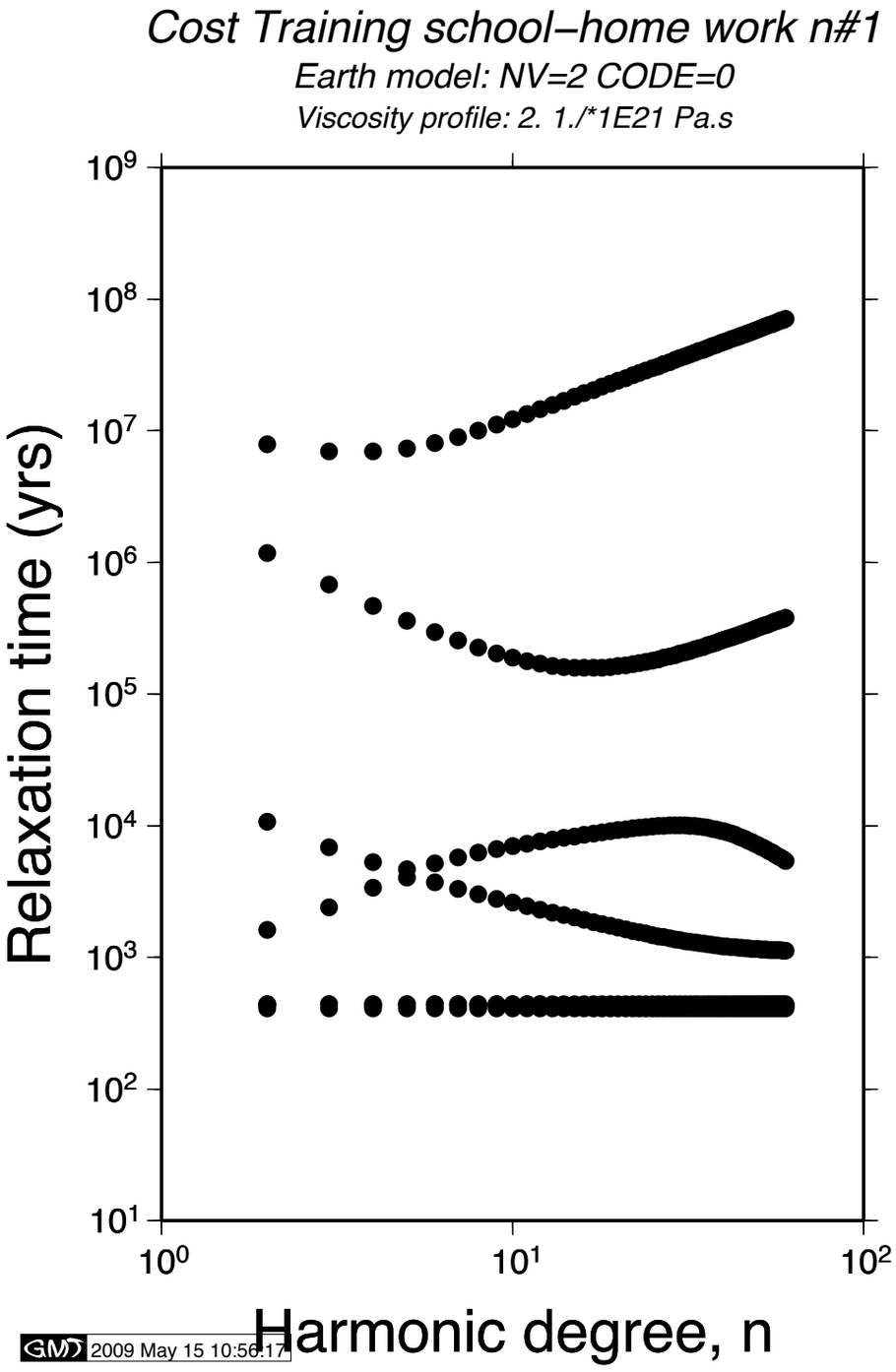
A viscoelastic **incompressible** model with no density inversions is ***asymptotically stable***.

It is more convenient (and physically intuitive) to consider the Love numbers in response to an Heaviside step function *(this corresponds to a creep experiment ...).*

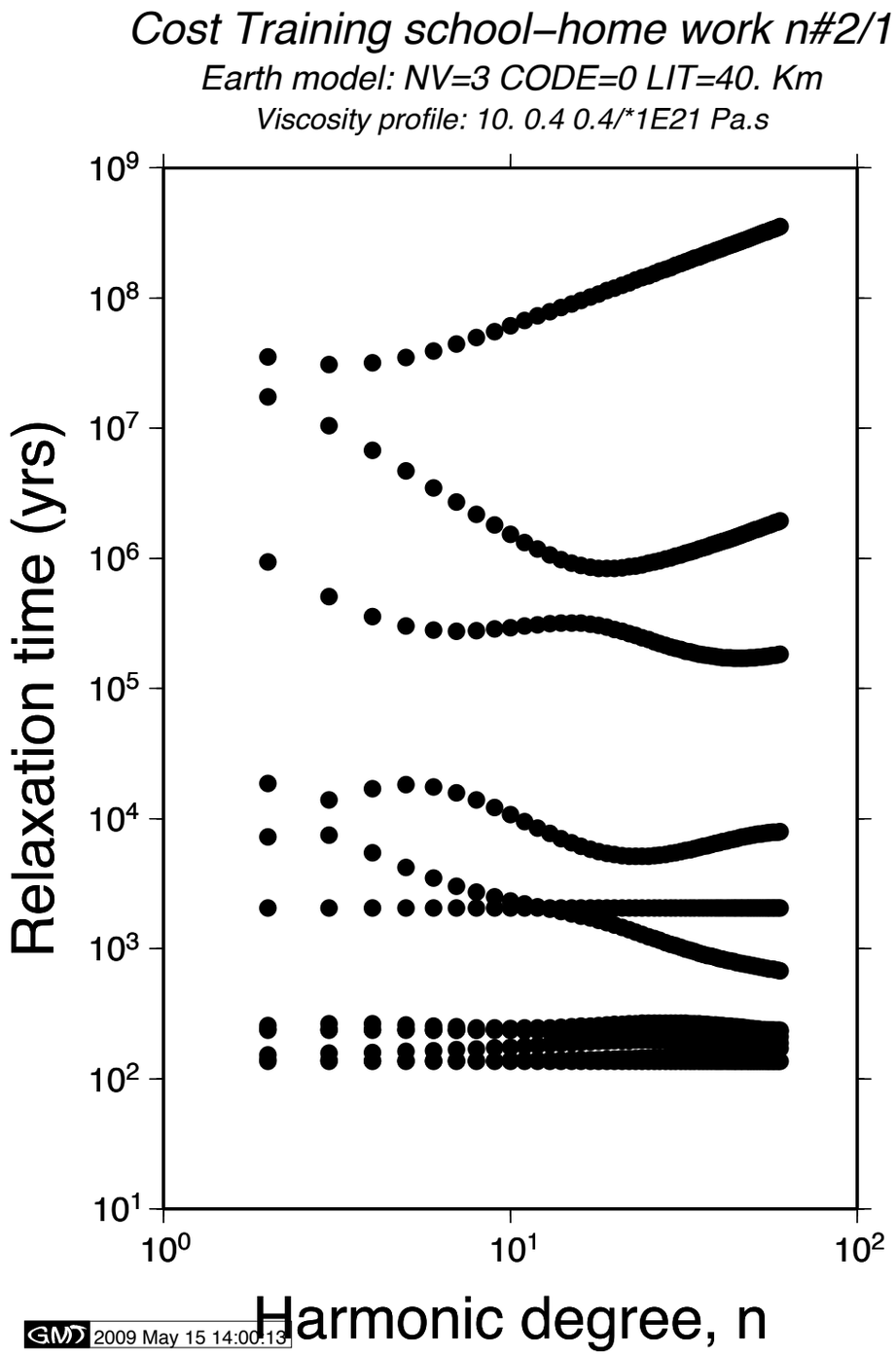
The Heaviside Love number has the following complex "multi-exponential form", a superposition of exponentially decaying terms:

$$h(t) = H(t)\left[h_e - \sum_{k=1}^{M} h_k \tau_k (1 - e^{-t/\tau_k})\right]$$

Love numbers (load-deformation coefficients)

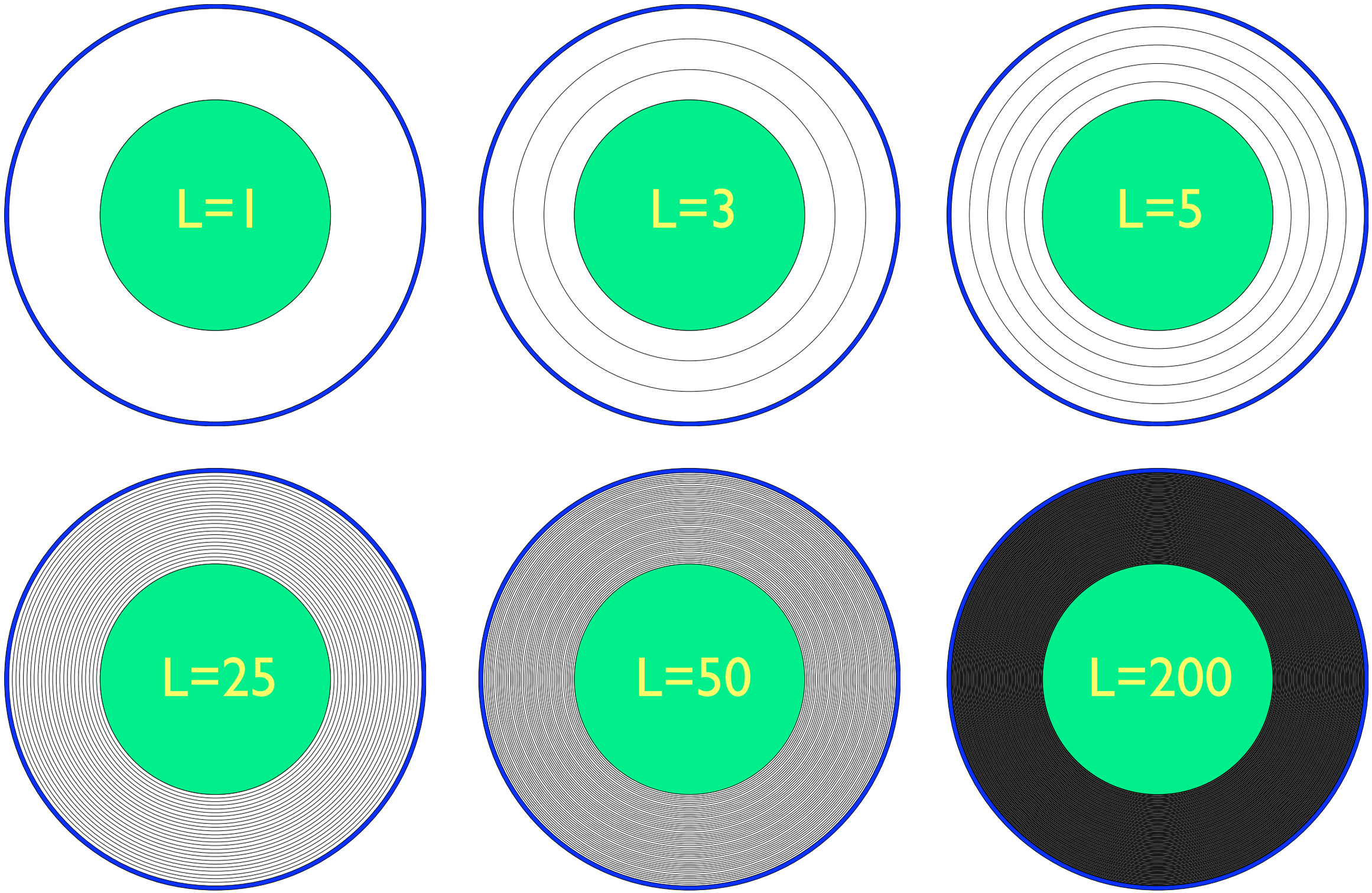


(L=2)



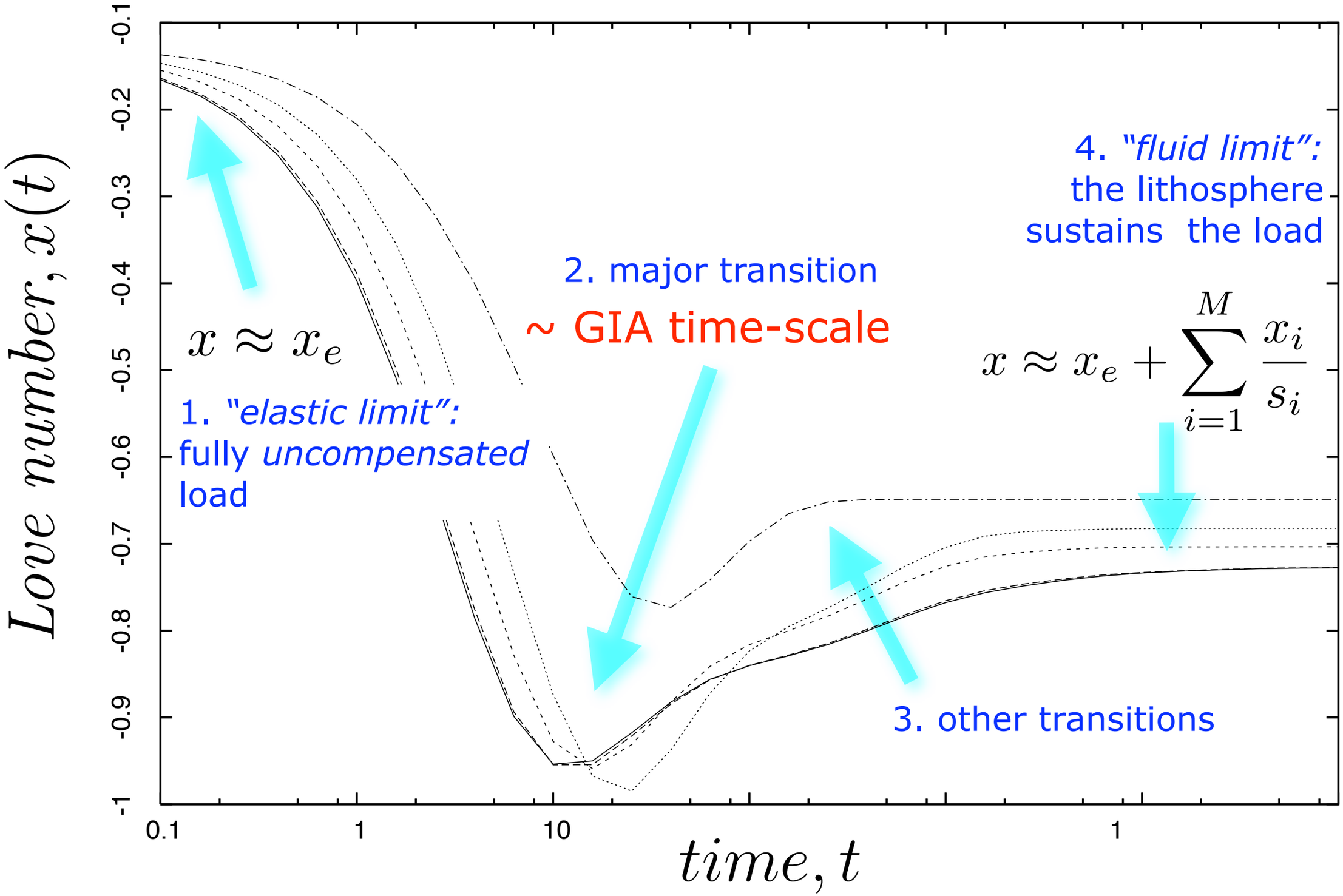
(L=3)

Love numbers (load-deformation coefficients)



- Heaviside Love numbers

$$x(t) = H(t)\left[x_e + \sum_{i=1}^{M} \frac{x_i}{s_i}(1 - e^{s_i t})\right]$$



Outline

1). Basic things about Rheology

2). “Normal modes” (Spectra, Love numbers...)

3). “Loading problems” (recent PGR benchmark activity)

4). Special topic: non-traditional computation of Love numbers

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November 7–11, 2011

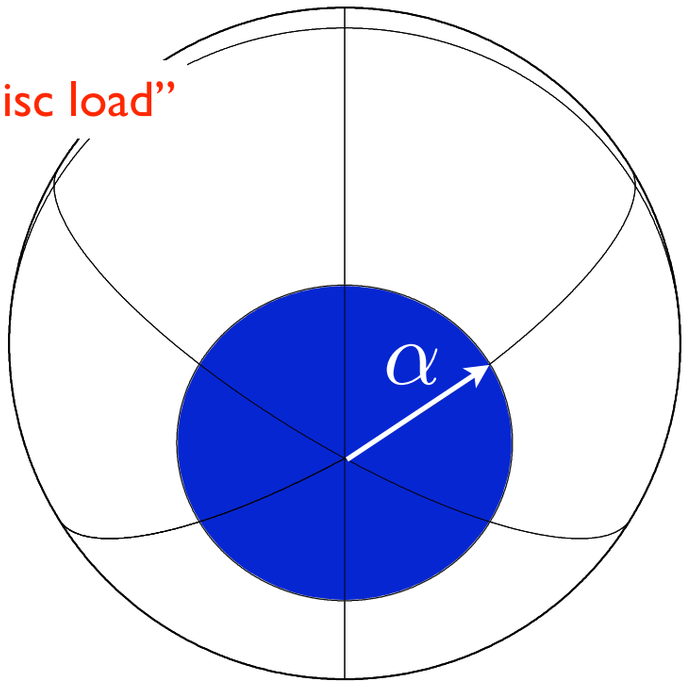
Northern hemisphere glaciation during the last ice ages. The set up of 3 to 4 km thick ice sheets caused a sea level lowering of about 120 m.

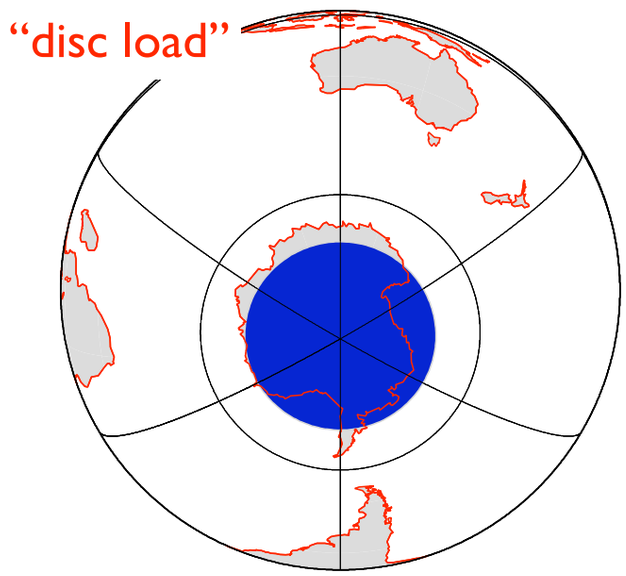
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M
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LA
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Winter sea ice

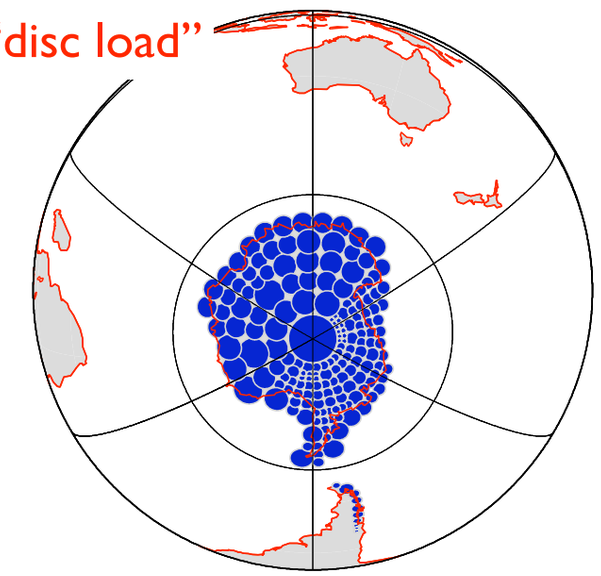
Ice loads northern hemisphere LGM

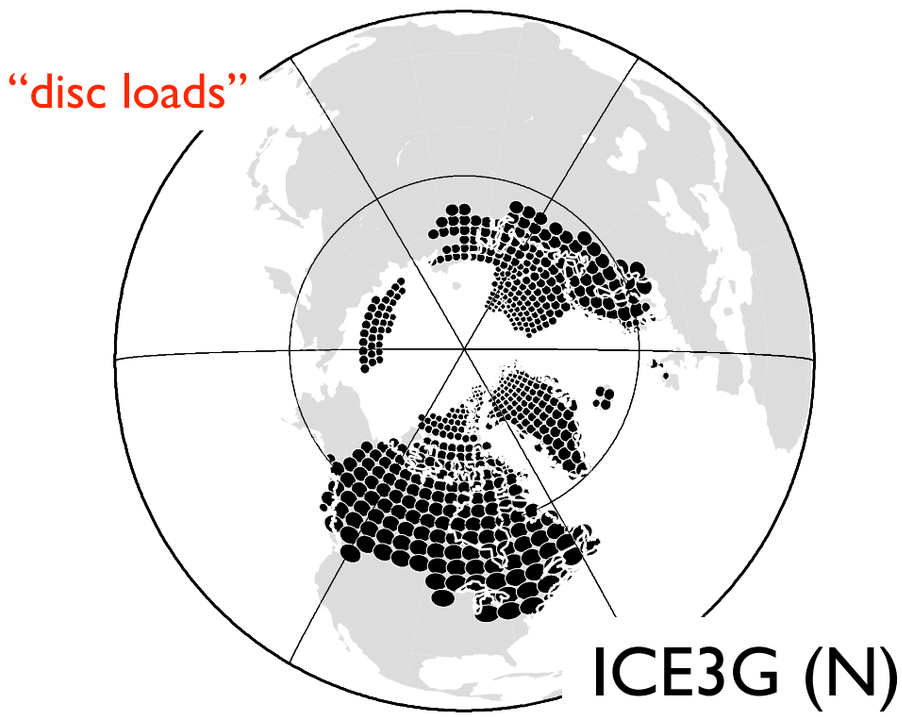
Ice loads

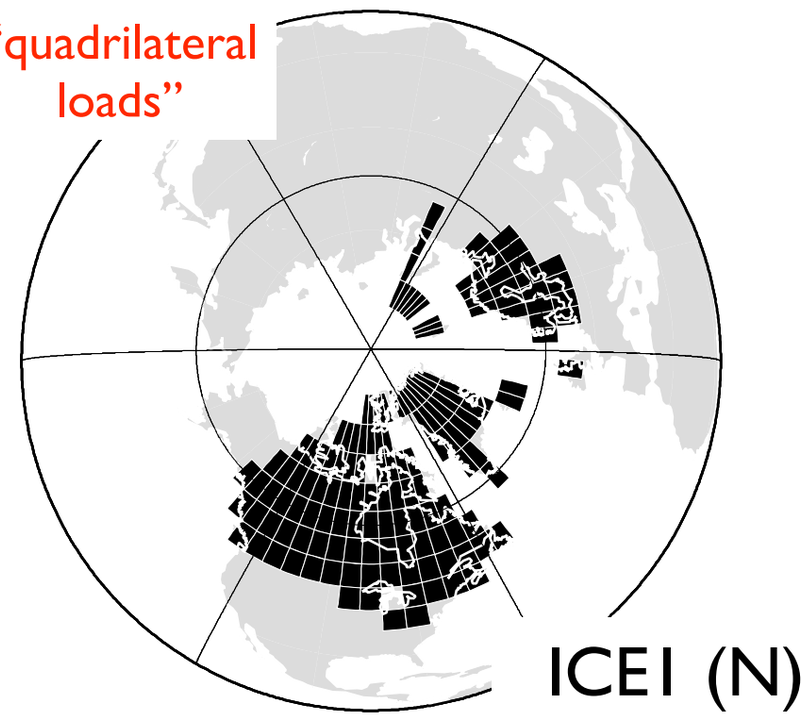
examples of ice loads





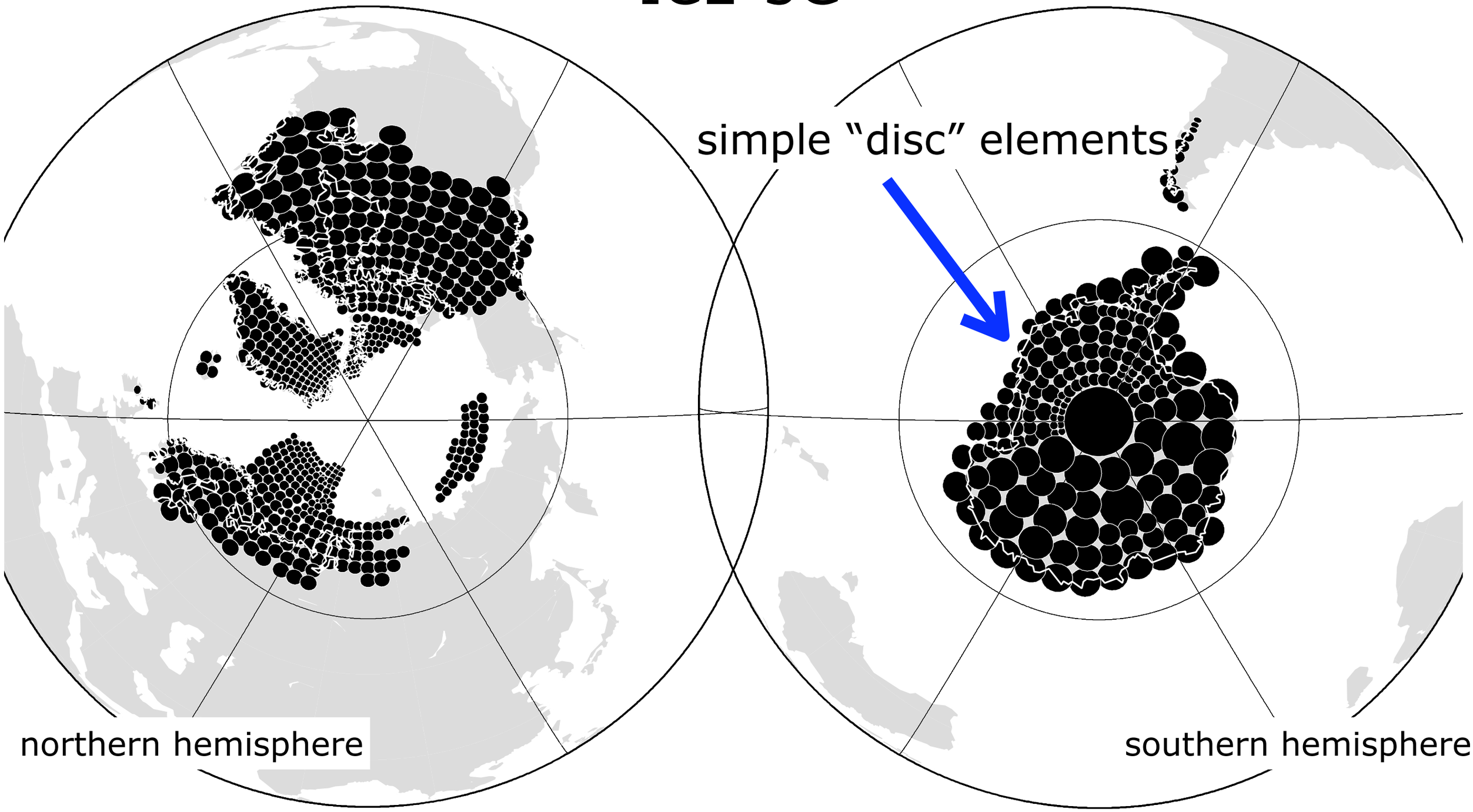




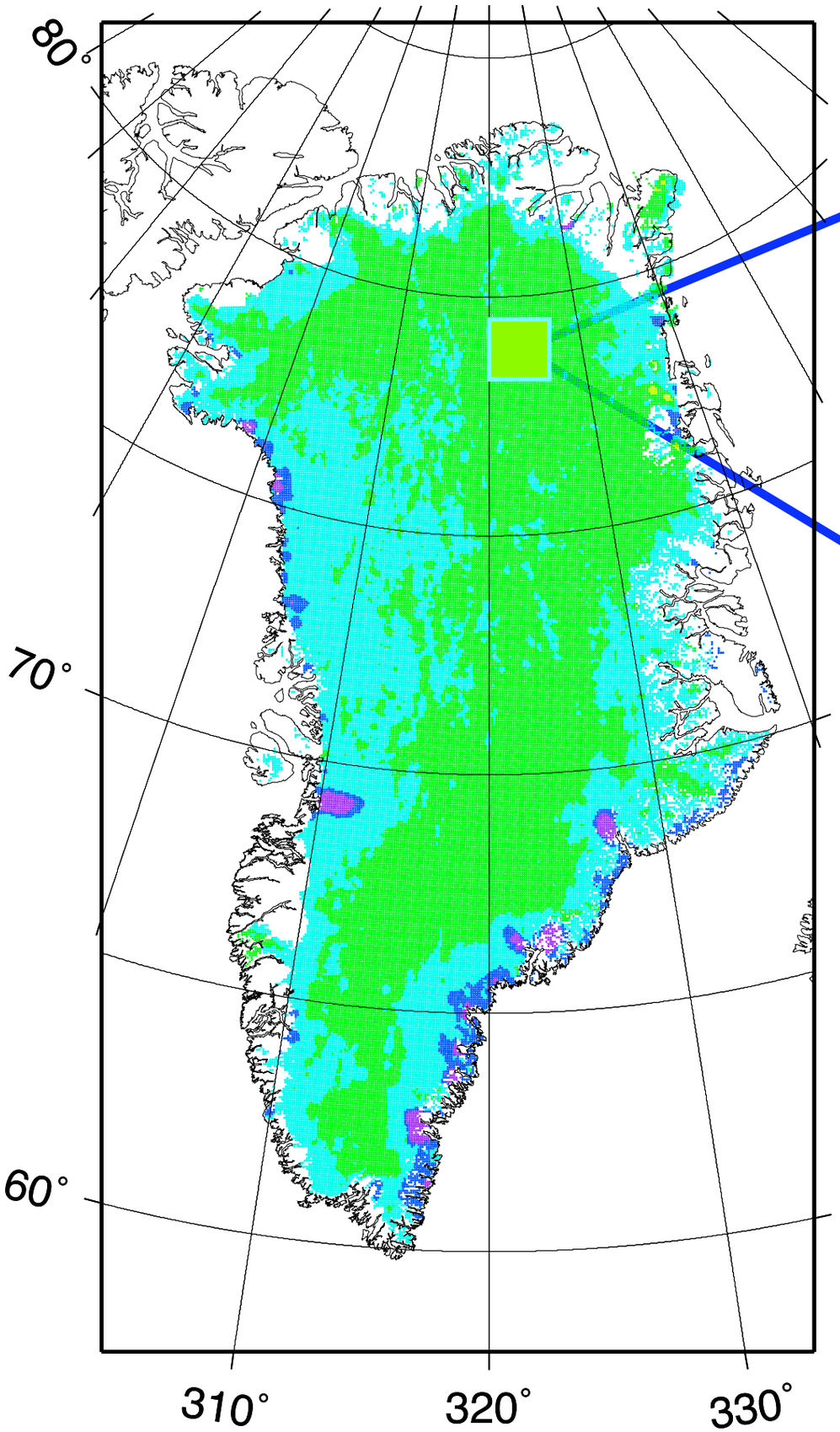


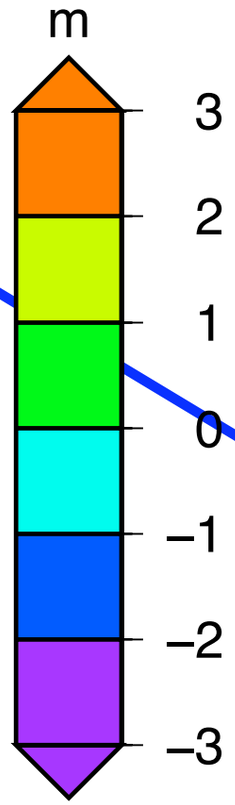
examples of ice loads

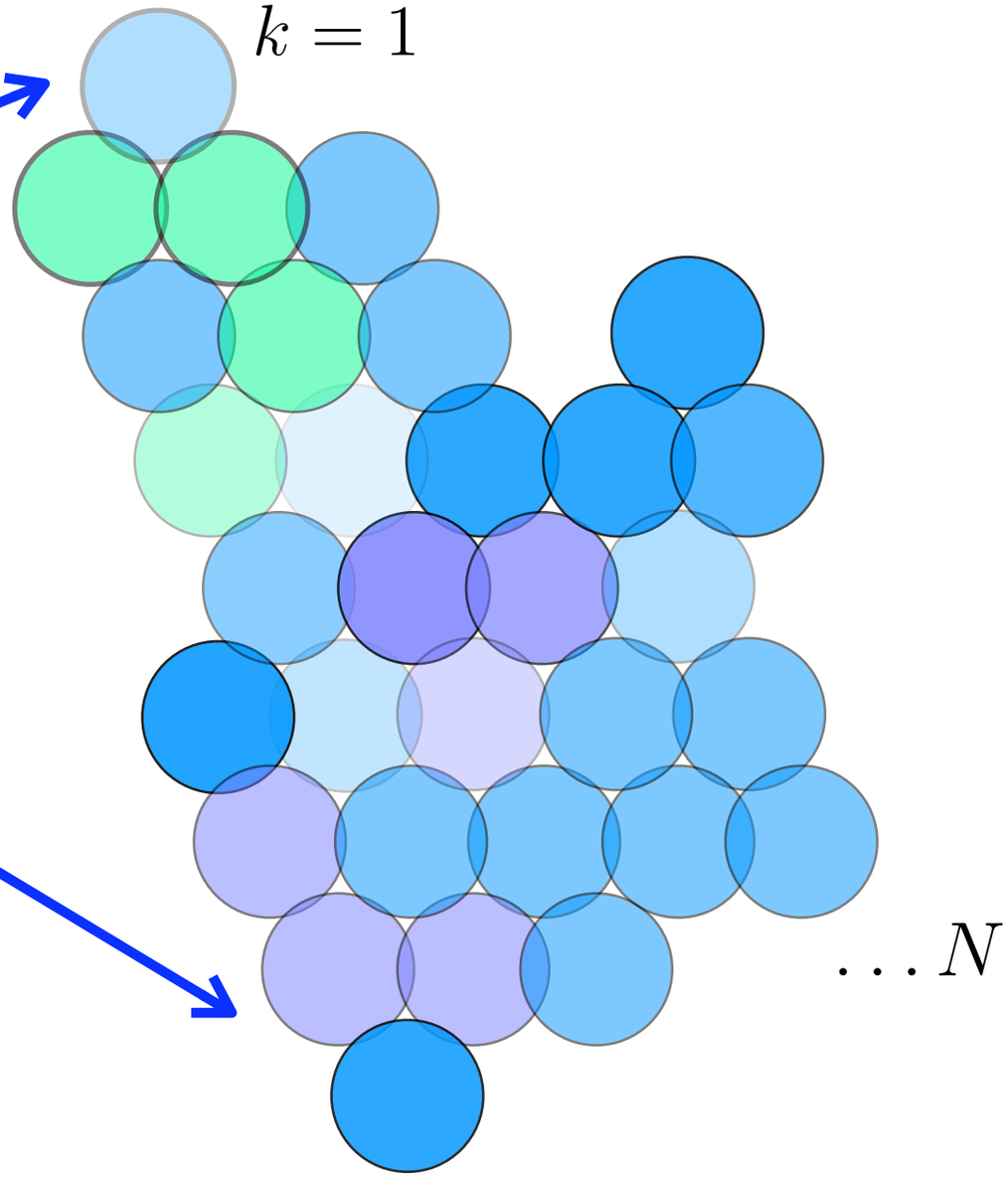
ICE-3G



Tushingham, A. M., Peltier, W. R., 1991.
Ice-3G: a new global model of late Pleistocene deglaciation based upon geophysical prediction of post--glacial sea level change. Journal of Geophysical Research 96, 4497-4523.







Number of elements: $N \approx 70 \times 10^3$

Size of each element: $\alpha_k \approx 0.025°$

Thickness is variable: $\Delta h = F(k)$

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A benchmark study for glacial isostatic adjustment codes

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SUMMARY

The study of glacial isostatic adjustment (GIA) is gaining an increasingly important role within the geophysical community. Understanding the response of the Earth to loading is crucial in various contexts, ranging from the interpretation of modern satellite geodetic measurements (e.g. GRACE and GOCE) to the projections of future sea level trends in response to climate change. Modern modelling approaches to GIA are based on various techniques that range

Benchmark codes, contributors and tests summary

Table 1. Top: participating WG4 COST Action ES0701 members and external contributors (starred), with notes about the independently developed codes and methods. Bottom: contributions to the benchmark tests so far. (Here abbreviation TX/Y denotes test 'X' of Test Class 'Y'.)

Acronym	Code	Author or user	Short description of method and notes
Bl	ABAQUS	B. Lund	Finite elements (FE) (Abaqus 2007); Flat-Earth approximation
Gs	TABOO	G. Spada	Viscolastic Normal Modes (VNM). Fortran code and User Guide available from Gs or from the Samizdat Press (http://samizdat.mines.edu/taboo/)
Gs	PMTF	G. Spada	A Fortran program that computes the polar motion transfer function using the VNM method (available from Gs)
Gsa	ALMA	G. Spada	Post-Widder method (Spada & Boschi 2006; Spada 2008). Fortran code available from http://www.fis.uniurb.it/spada/ALMA_minipage.html
Pg*	ABAQUS	P. Gasperini	Finite elements (FE) (Abaqus 2007)
Rr	FastLove-HiDeg	R. E. M. Riva	VNM (Riva & Vermeersen 2002)
Vb	MHPLove	V. R. Barletta	A Mathematica^{TM} Mathematica 4.1 2001 program for computing high precision Love numbers
Vk	VILMA	V. Klemann	Spectral-finite elements (SFE) Martinec (2000)
Zm	VEENT	Z. Martinec	VNM (Martinec & Wolf 1998), Fortran code available from Zm

Test	Bl	Gs	Gsa	Pg*	Rr	Vb	Vk	Zm	Meaning of test
T1/1		x			x	x		x	Isostatic relaxation times
T2/1		x			x	x		x	Loading Love numbers
T3/1		x			x	x		x	Tidal Love numbers
T4/1		x				x			Polar motion transfer function
T5/1		x	x				x		Time-domain Love numbers
T6/1		x	x		x		x		Love numbers of multistratified model
T7/1		x	x		x	x	x		Degree-1 Love numbers
T1/2	x	x		x			x	x	Geodetic quantities
T2/2	x	x		x			x		Rates of geodetic quantities
T3/2		x				x			Polar motion and LOD

Some important formulas for finite sized loads (I)

1. Surface load
Fixed margins: the load is "factorized"

$$L(\theta, t) = \sigma(\theta) f(t)$$

2. "Load function"
depends on the load geometry

$$\sigma(\theta) = \sum_{n=n_{min}}^{n_{max}} \sigma_n P_n(\cos\theta)$$

$$\left\{ \begin{array}{c} \bar{h} \\ \bar{l} \\ \bar{k} \end{array} \right\}^L (t) \equiv \left\{ \begin{array}{c} h^L(t) \\ l^L(t) \\ \delta(t) + k^L(t) \end{array} \right\} * f(t)$$

3. Convolved
Love numbers

Some important formulas for finite sized loads (II)

4. An important case: Heaviside Love numbers

$$\left\{ \begin{array}{c} \bar{h} \\ \bar{l} \\ \bar{k} \end{array} \right\}^L (t) = \left\{ \begin{array}{c} h_e^L \\ l_e^L \\ 1 + k_e^L \end{array} \right\} - \sum_{j=1}^{M} \left\{ \begin{array}{c} h_j \\ l_j \\ k_j \end{array} \right\} \frac{1 - \mathrm{e}^{s_j t}}{s_j}, \qquad t \geq 0$$

5. Final response:

$$\left\{ \begin{array}{c} U \\ V \\ N \end{array} \right\} (\theta, t) = \frac{3}{\bar{\rho}_e} \sum_{n=n_{min}}^{n_{max}} \left\{ \begin{array}{c} \bar{h} \\ \bar{l} \\ \bar{k} \end{array} \right\}^L (t) \; \frac{\sigma_n}{2n+1} \left\{ \begin{array}{c} 1 \\ \frac{\partial}{\partial \theta} \\ 1 \end{array} \right\} P_n(\cos\theta)$$

$$\left\{ \begin{array}{c} U \\ V \\ N \end{array} \right\} (\theta, t) = \left\{ \begin{array}{c} \text{Vertical displacement} \\ \text{Horizontal displacement} \\ \text{Geoid displacement} \end{array} \right\}$$

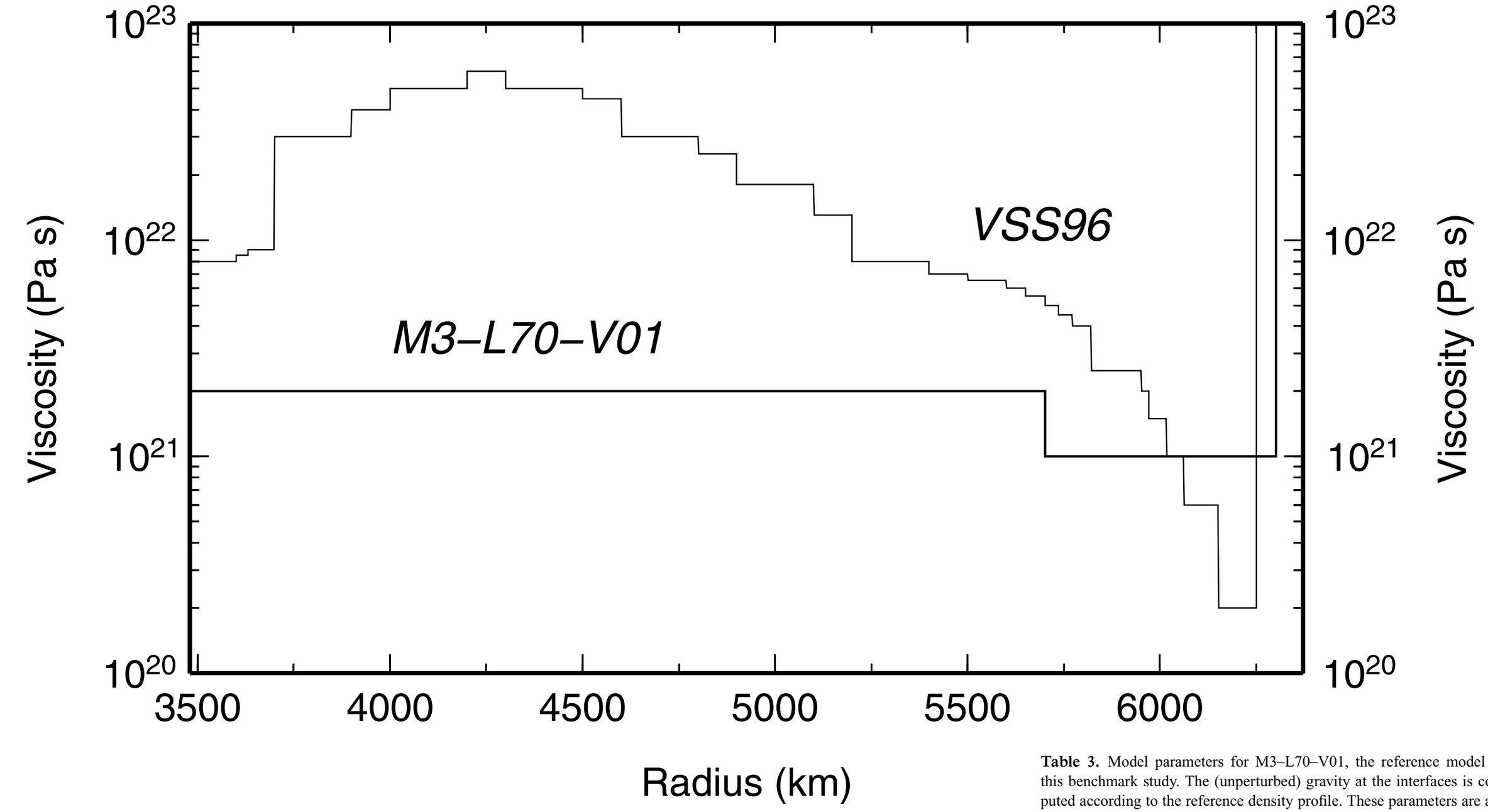
Geometry of the ice loads: point-like, disc and cap

Table 4. Description of the geometry of the ice loads. For all loads, the ice density is $\rho_i = 931$ kg m^{-3}. The reference value for Earth radius a and the definition of $T_n(x)$, the Chebyshev polynomials of the second kind, are given in Table 2. Here h and α denote the ice thickness and the load half-extension, respectively, while m_δ is the mass of the point load. The Legendre coefficients σ_n are adopted from Spada (2003). Reference numerical values for h, α and m_δ suggested for the spatial domain tests are also given. For Test 2/3, the colatitude and longitude of the centroid of the load geometries are $\theta_c = 25^\circ$ and $\lambda_c = 75^\circ$.

Shape	Surface mass density of load, $\sigma(\theta)$	Legendre coefficients, $\sigma_n (n \geq 1)$	Numerical values
Cap	$\sigma^c(\theta) = \rho_i h \begin{cases} \sqrt{\dfrac{\cos\theta - \cos\alpha}{1-\cos\alpha}}, & 0 \leq \theta \leq \alpha, \\ 0, & \theta > \alpha. \end{cases}$	$-\dfrac{\rho_i h}{4(1-\cos\alpha)}\left[\dfrac{T_{n+1}(\alpha) - T_{n+2}(\alpha)}{n + {}^3\!/_2} - \dfrac{T_{n-1}(\alpha) - T_n(\alpha)}{n - {}^1\!/_2}\right]$	$h = 1500\,m$ $\alpha = 10^\circ$
Disc	$\sigma^d(\theta) = \rho_i h \begin{cases} 1, & 0 \leq \theta \leq \alpha, \\ 0, & \theta > \alpha. \end{cases}$	$\dfrac{\rho_i h}{2}\left[-P_{n+1}(\cos\alpha) + P_{n-1}(\cos\alpha)\right]$	$h = 1000\,m$ $\alpha = 10^\circ$
Point	$\sigma^\delta(\theta) = \dfrac{m_\delta}{2\pi a^2}\delta(1-\cos\theta)$	$\dfrac{m_\delta}{4\pi a^2}(2n+1)$	$m_\delta = 10^{18}$ kg

$$L(\theta, t) = \sigma(\theta) f(t) \qquad \sigma(\theta) = \sum_{n=n_{min}}^{n_{max}} \sigma_n P_n(\cos\theta)$$

Benchmark viscosity profiles: coarse and multilayered



GMT 2010 Apr 15 21:40:12

Table 3. Model parameters for M3–L70–V01, the reference model for this benchmark study. The (unperturbed) gravity at the interfaces is computed according to the reference density profile. These parameters are also adopted in model M3–L70–V01f, the flat-Earth variant of M3–L70–V01 implemented in the FE scheme of Bl (see Section 4.2.3).

Radius (km)	Density (kg m^{-3})	Shear modulus ($\times 10^{11}$ Pa)	Viscosity ($\times 10^{21}$ Pa s)	Gravity (m s^{-2})
6371	3037	0.50605	∞	9.815
6301	3438	0.70363	1	9.854
5951	3871	1.05490	1	9.978
5701	4978	2.28340	2	10.024
3480	10750	0	0	10.457

Relaxation spectrum for model M3-L70-V01: 12 branches

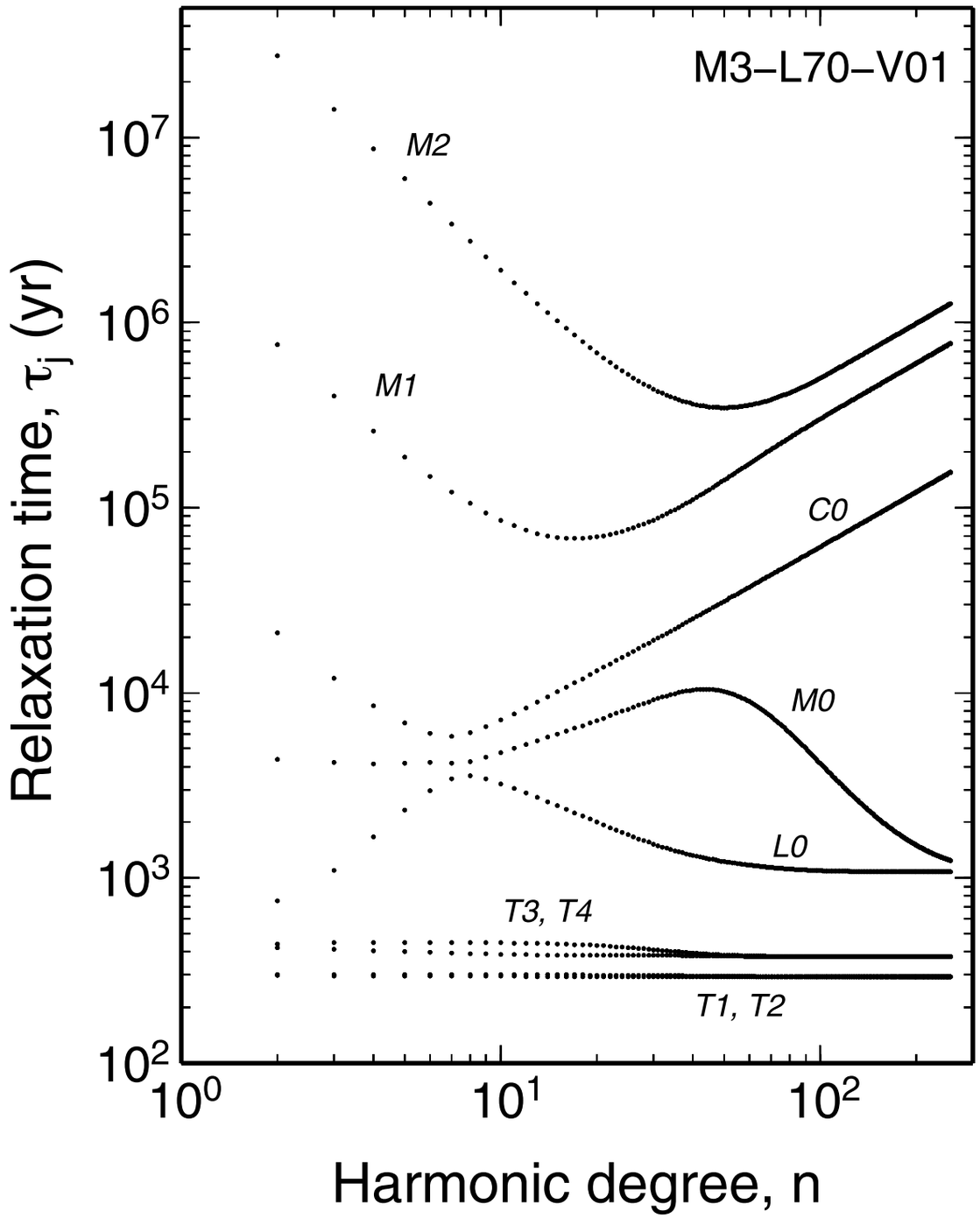
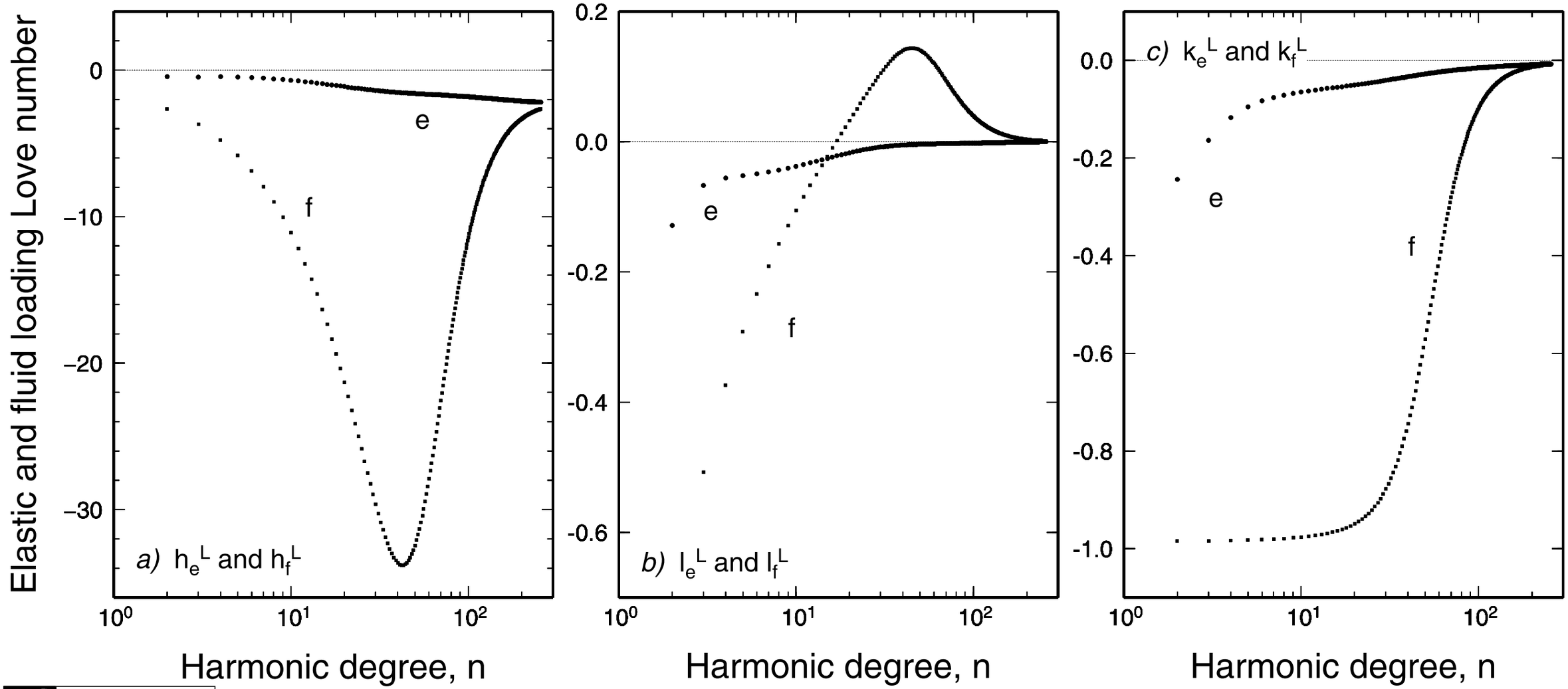


Figure 2. Spectrum of the isostatic relaxation times τ_j, ($j = 1, \ldots, M$), expressed in years, for model M3–L70–V01 in the range of degrees $2 \leq n \leq 256$, according to data sets provided by Vb, Rr, Gs and Zm (Test 1/1). The number of modes is $M = 9$, branches are labelled according to usual conventions.

Elastic and fluid load-deformation coefficients *(h, l, k)*



GMD 2010 Jun 18 09:35:02

Figure 3. Elastic (e, circles) and fluid (f, squares) loading Love numbers for model M3–L70–V01 as a function of harmonic degree (Test 2/1), computed according to contributors Vb, Rr and Gs. Vertical, horizontal and potential Love numbers are shown in frames (a), (b) and (c), respectively.

Viscoelastic residues for model M3-L70-V01: 12 branches

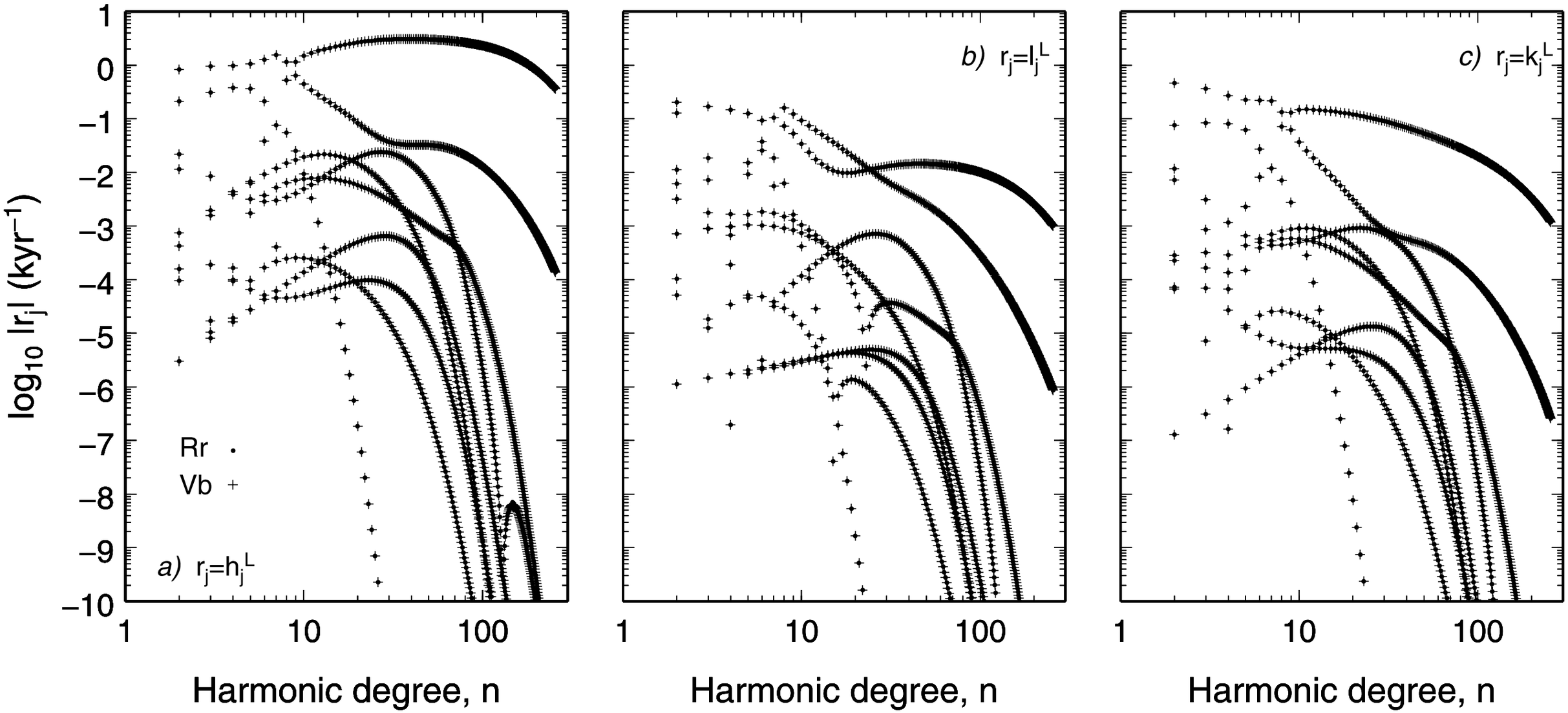
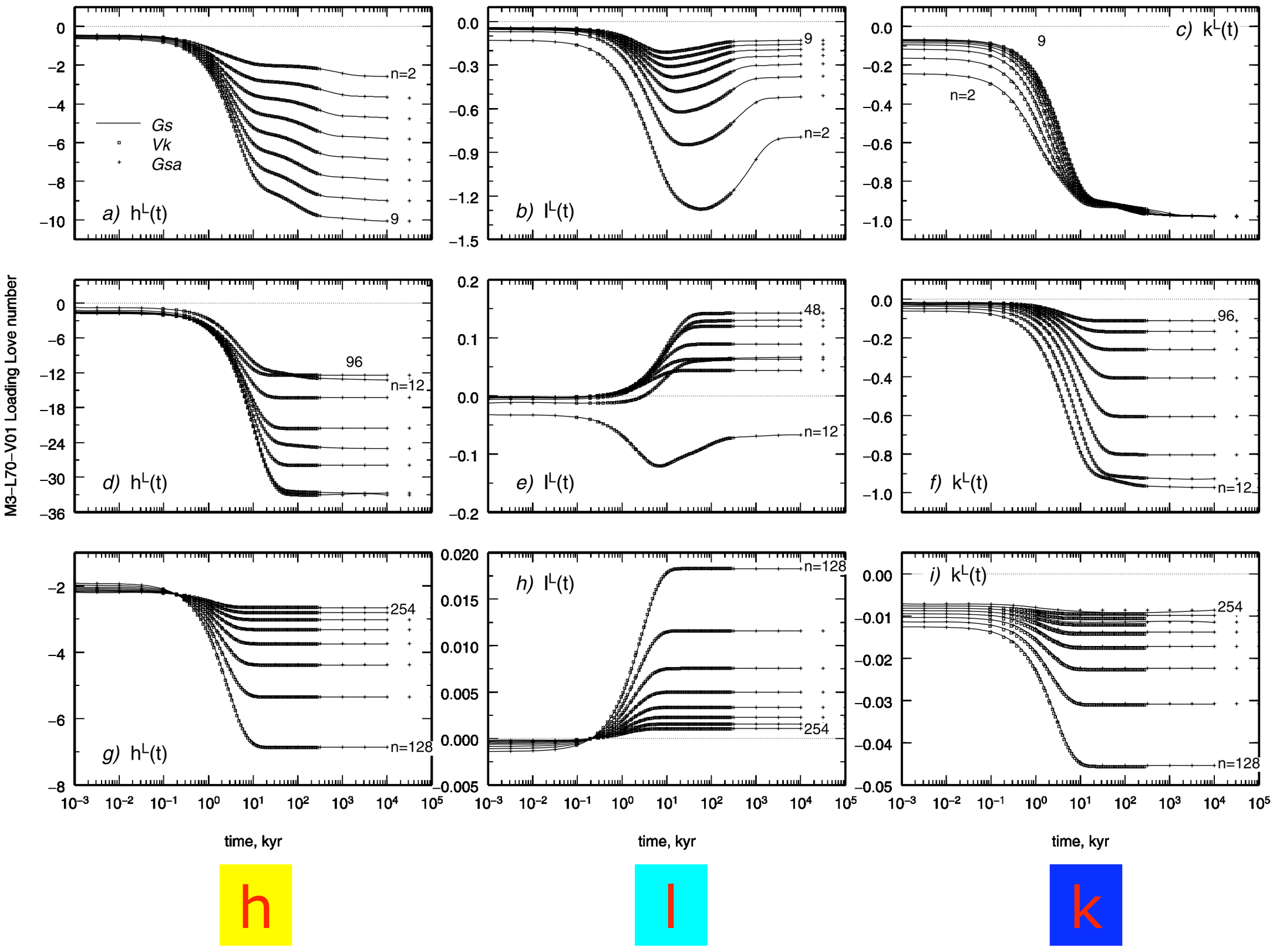


Figure 4. Comparison between absolute values of residues of the loading Love numbers computed for model M3–L70–V01 by Rr and Vb (Test 2/1). Smoothness of the nine branches with varying *n* may indicate that Love number formalism is correctly implemented.

“Heaviside” load-deformation coefficients: *Vk, Gs, Gsa*



Degree ONE Love numbers for model M3-L70-V01

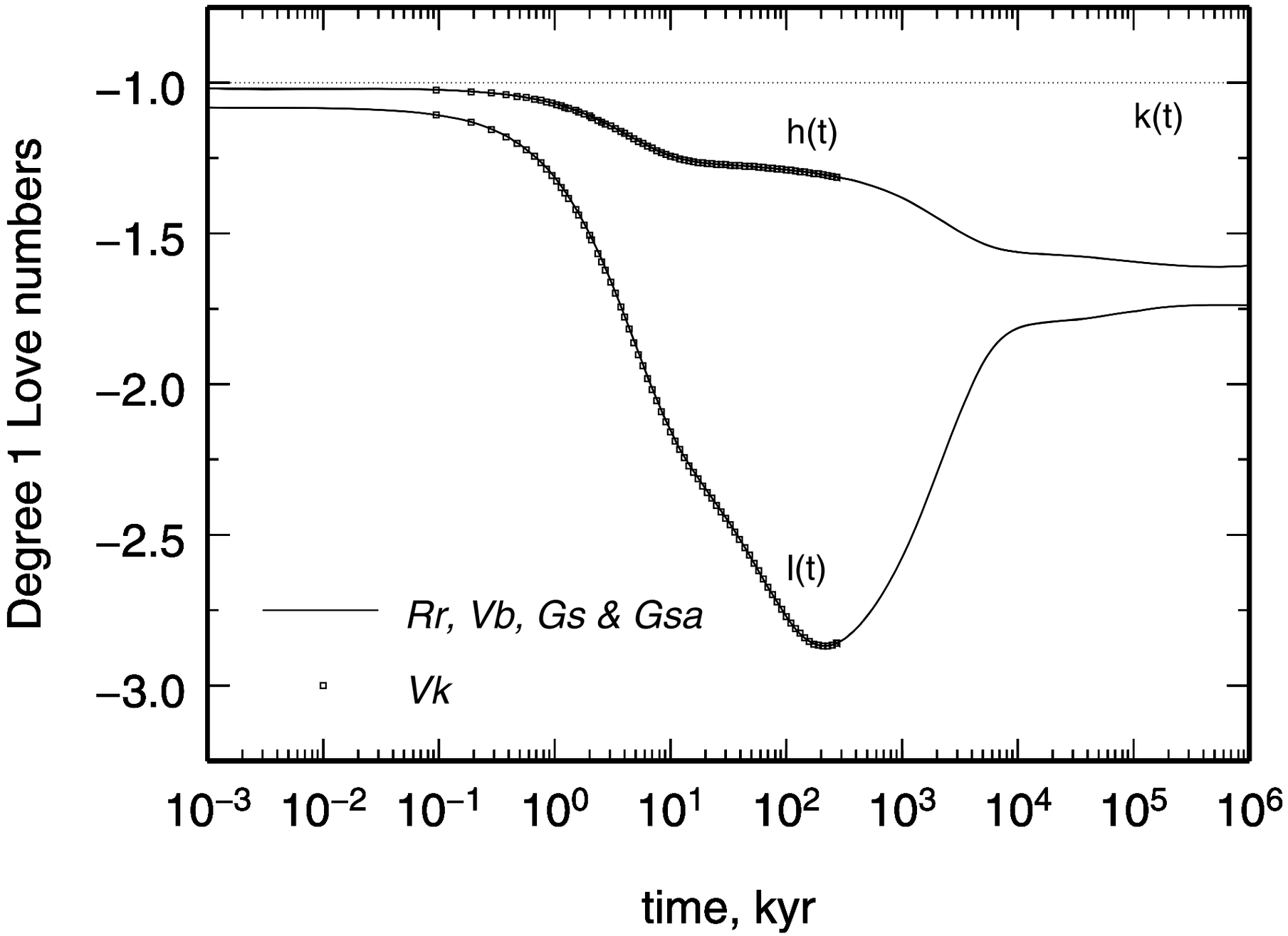


Figure 8. Time-domain loading Love numbers of degree $n = 1$ for Test 7/1 obtained by contributors Rr, Vb, Gs (by VNM), Gsa (by Post–Widder formula) and Vk (by SFE, squares). Note that $k^L(t) = -1$ (dashed line).

Response to a Heaviside loading, model M3-L70-V01 (I)

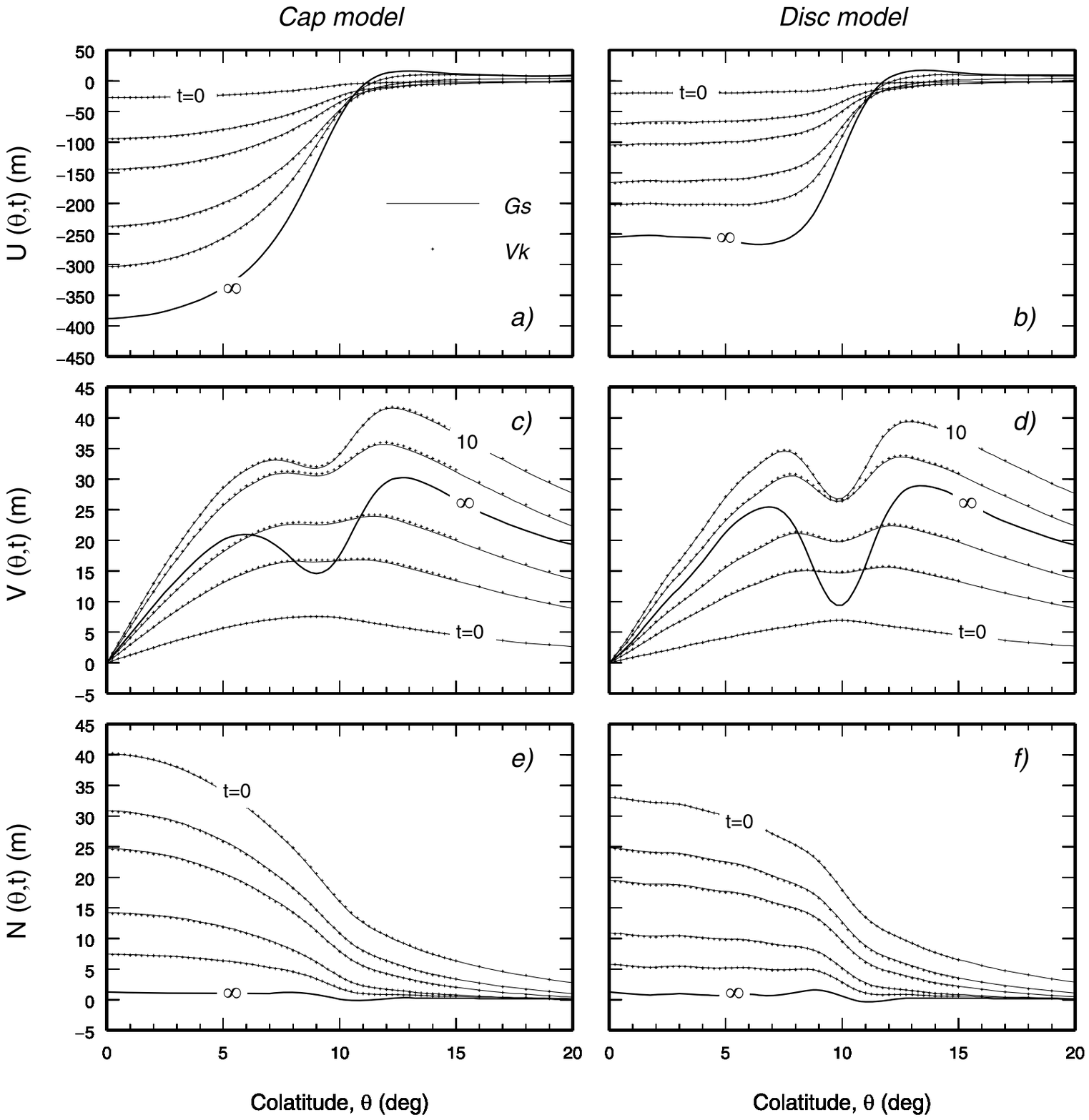


Figure 9. Vertical, horizontal and geoid displacements at the Earth's surface (Test 1/2) according to Gs (VNM method, solid lines) and Vk (SFE, dots) for a cap ice-sheet model (left-hand side) and a disc (right-hand side) imposed at time $t = 0$. Symbol '∞' stands for $t = 10^3$ kyr for Gs and corresponds to isostatic equilibrium, this datum is not available for Vk.

Response to a Heaviside loading, model M3-L70-V01 (II)

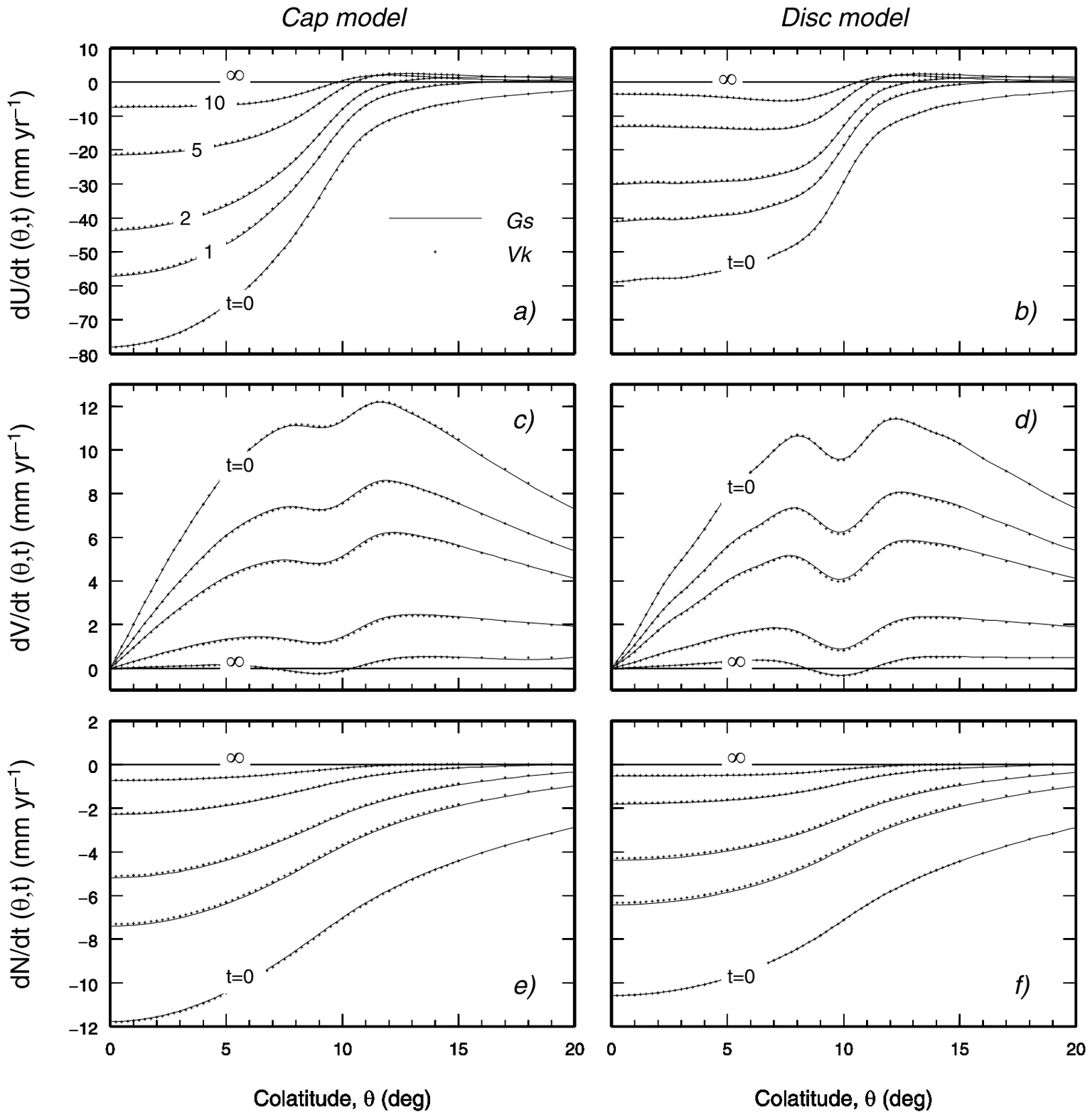


Figure 10. Rates of vertical, horizontal and geoid displacements at the Earth's surface (Test 2/2) according to Gs (VNM method, solid lines) and Vk (SFE, dots) for a cap ice-sheet model (left-hand side) and a disc (right-hand side) imposed at time $t = 0$. Symbol '∞' marking the horizontal lines in each plot stands for $t = 10^3$ kyr.

Response to a Heaviside load, model M3-L70-V01 (II)

Table 10. Comparison between vertical displacements $U(\theta, t)$ independently computed by Vk, Gs and Zm for Test 1/2 (cap load model). Numbers in parentheses following the contributor abbreviation is time in units of kyr. Units for U, V and N are metres throughout. The figures are obtained by truncating the outputs of codes at the centimetre level. Superscripts a and b indicate that the corresponding quantities are computed by Vk at $\theta = 0.01^\circ$ and $\theta = 179.99^\circ$, respectively. Symbol '∞' stands for $t = 10^3$ kyr for Gs. The corresponding values are not available for Vk and Zm, since the computations would require an exceedingly long CPU time.

θ (deg)	t (kyr)	Vk, U (m)	Gs, U (m)	Zm, U (m)	θ (deg)	t (kyr)	Vk, U (m)	Gs, U (m)	Zm, U (m)
0°	0	-27.81^a	−27.77	−27.77	45°	0	0.60	0.61	0.60
	1	-94.49^a	−94.40	−94.42		1	2.28	2.27	2.27
	5	-237.58^a	−237.49	−237.50		5	4.37	4.35	4.35
	10	-303.03^a	−303.01	−302.99		10	4.63	4.61	4.62
	∞	...	−388.11	...		∞	...	6.20	...
5°	0	−23.65	−23.64	−23.65	90°	0	0.51	0.51	0.51
	1	−79.59	−79.53	−79.55		1	1.01	1.00	1.01
	5	−199.62	−199.59	−199.60		5	1.28	1.27	1.28
	10	−256.98	−257.05	−257.03		10	1.46	1.45	1.46
	∞	...	−338.30	...		∞	...	1.70	...
10°	0	−7.49	−7.33	−7.38	120°	0	−0.14	−0.14	−0.14
	1	−25.25	−24.80	−24.92		1	−0.42	−0.42	−0.42
	5	−48.88	−47.73	−48.04		5	−0.64	−0.63	−0.63
	10	−50.94	−49.35	−49.77		10	−0.72	−0.71	−0.71
	∞	...	−59.24	...		∞	...	−0.95	...
20°	0	−1.20	−1.20	−1.20	180°	0	-0.85^b	−0.85	−0.85
	1	−1.72	−1.72	−1.72		1	-1.82^b	−1.82	−1.82
	5	3.87	3.85	3.85		5	-2.67^b	−2.66	−2.66
	10	7.07	7.03	7.04		10	-2.97^b	−2.97	−2.97
	∞	...	8.55	...		∞	...	−3.58	...

Outline

1). Basic things about Rheology

2). “Normal modes” (Spectra, Love numbers...)

3). “Loading problems” (recent PGR benchmark activity)

4). Special topic: non-traditional computation of Love numbers

Charles University in Prague, Department of Geophysics

November 7–11, 2011

Recent work on viscoelastic normal modes

(with some applications)

Giorgio Spada

DiSBeF, Urbino University "Carlo Bo"

http://www.fis.uniurb.it/spada

giorgio.spada@gmail.com

Outline:

New ideas about the computation of the Load-deformation coefficients for a *"Generalized Maxwell Earth"*:

1) (condensed) theory, and

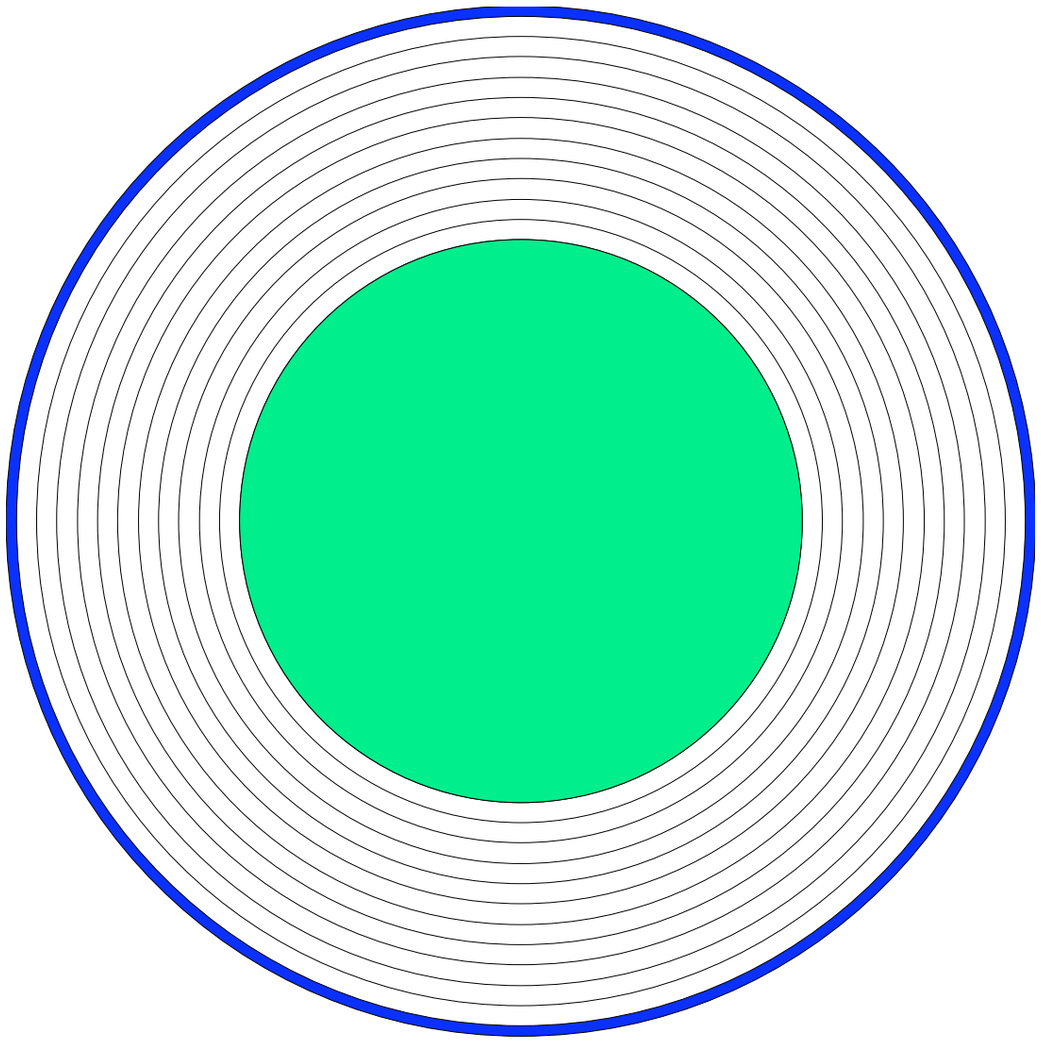
2) a few applications to the GIA problem.

Goals:

i) Computing (and understanding) Love numbers in ***extreme rheological situations*** (e. g., low-viscosity layers or complex rheologies),

ii) providing all WGs and the community with new, more accurate Love numbers (where required by data) and free software.

Spherically symmetric Earth model

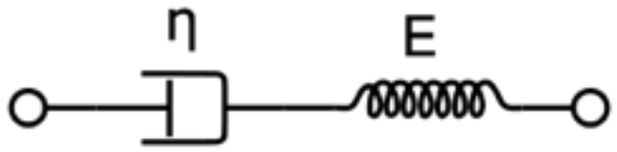


(elastic) Lithosphere

"L" mantle layers

Fluid core

Linear viscoelastic rheology (e. g., Maxwell)



Love numbers *(Love, 1909) are three non-dimensional numbers (h, l, k) that provide, at a given harmonic degree, vertical and horizontal displacement and incremental potential for an impulsive localized source of deformation.*

Love numbers by Normal Modes (1/2)

Love numbers at any degree* in the Laplace domain are:

$$(h, l, k)(s) = [P_x W J][P_b W J]^{-1} \overrightarrow{b}$$

where

$$W(s) = \prod_{j=L+1}^{1} Y_j(r_{j+1}, s) Y_j^{-1}(r_j, s)$$

propagates the solution, and $Y_j(r, s)$ is a fundamental matrix, that depends upon rheology upon the complex shear modulus:

$$\tilde{\mu} = \mu(s) = \frac{\mu s}{s + \mu/\eta} \qquad \text{(Maxwell)}$$

(*) See work of B. Vermeersen and others for all the other details.

Love numbers by Normal Modes (2/2)

The time-domain LNs are usually obtained by Laplace inversion formula:

$$x(t) = x_e\delta(t) + \frac{1}{2\pi i}\int_{Br} e^{st}\tilde{x}(s)ds, \quad t \geq 0,$$

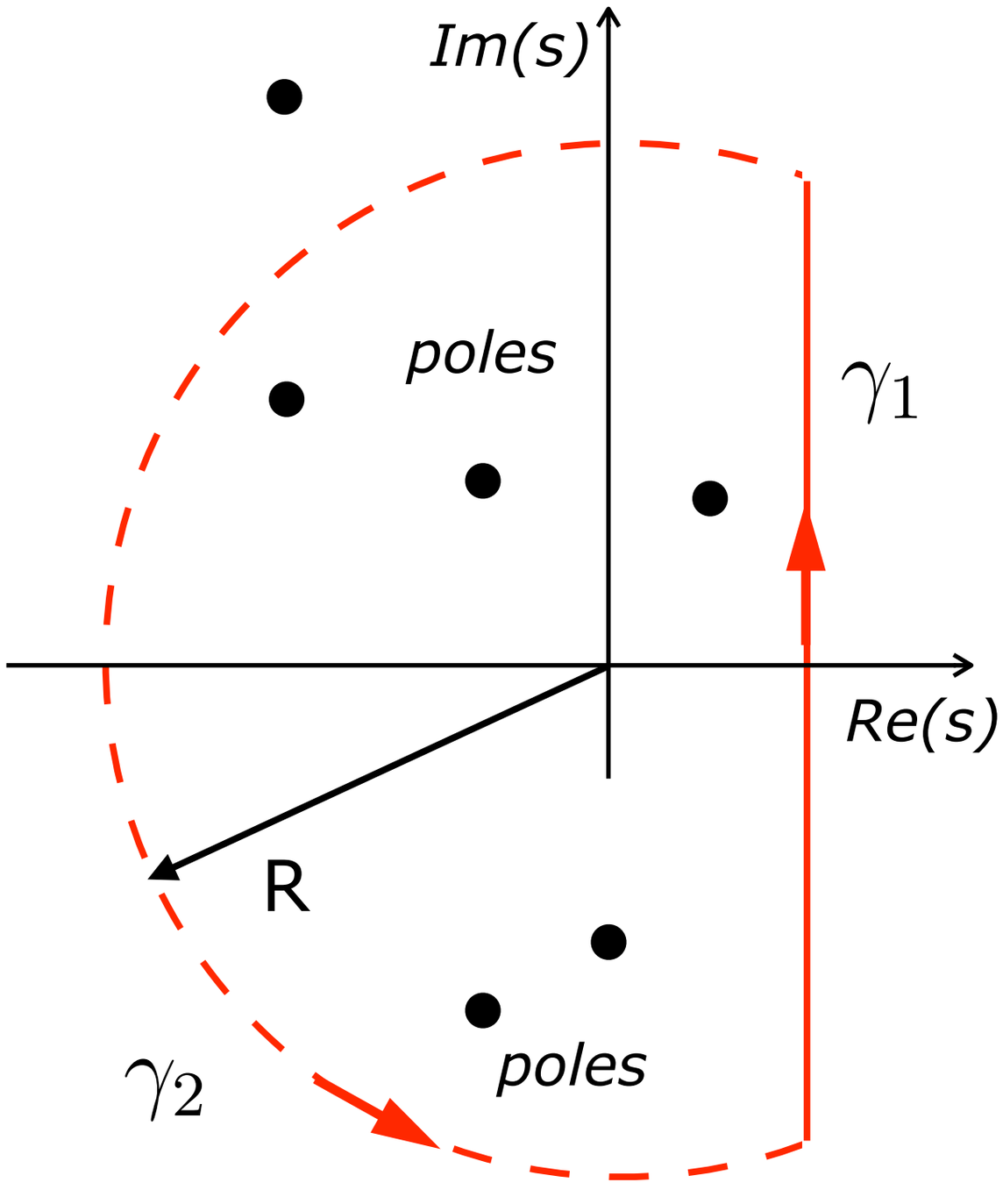
where *Br* is Bromwich path. By residues theorem we get:

$$x(t) = x_e\delta(t) + \sum_{i=1}^{M} x_i e^{s_i t}$$

where s_i are ROOTS of a secular equation and x_i the residues.

x_e and (x_i, s_i) are the "VISCOELASTIC NORMAL MODES"

an example



$$C = \gamma_1 \cup \gamma_2$$

- We observe that

$$\int_C = \int_{\gamma_1} + \int_{\gamma_2}$$

- Letting $R \mapsto \infty$ and using Jordan's lemma:

$$2\pi i \sum_k r_k(t) = \int_{Br} + \ 0$$

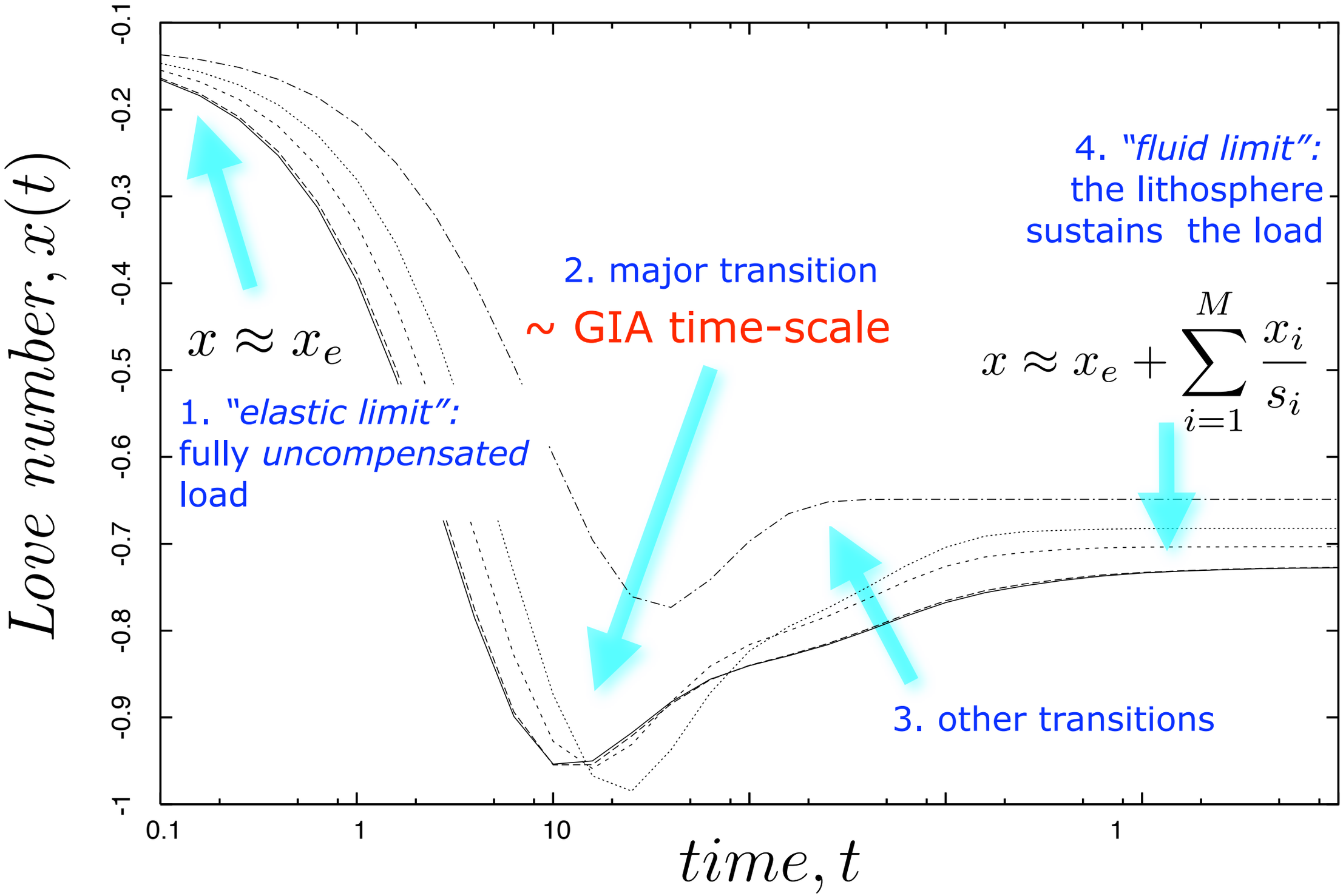
- Hence:

$$\mathcal{L}^{-1}\{\tilde{f}(s)\} = \sum_k r_k(t)$$

The time-domain response is multi-exponential...

- Heaviside Love numbers

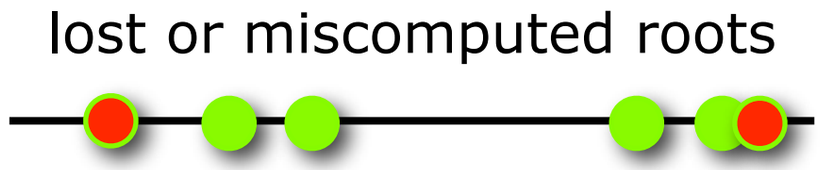
$$x(t) = H(t)\left[x_e + \sum_{i=1}^{M} \frac{x_i}{s_i}(1 - e^{s_i t})\right]$$



Problems with normal modes

1. We have no idea of *where* the most significant roots may be located..., therefore:

What if we *miss* one (or more) modes? ... or if we *miscompute* one ore more modes?



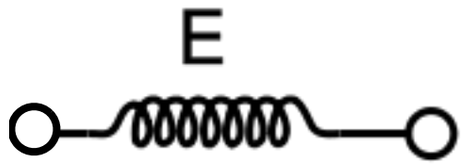
miscomputed roots will contaminate the whole solution

2. *algebraic complexity* increases with:

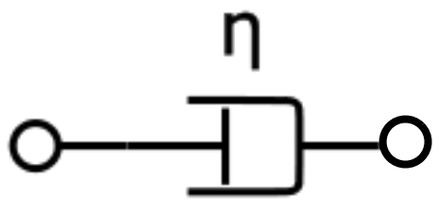
i) number of layers (*linearly*, for Maxwell rheology),
ii) rheological law,

3. changing the *rheological law* (e. g., from *Maxwell* to *Burgers*) demands a new, ad hoc code...

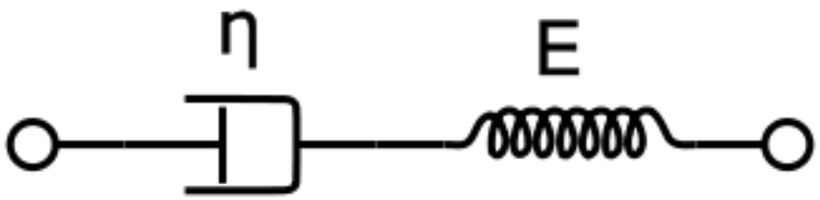
Linear viscoelastic rheological models



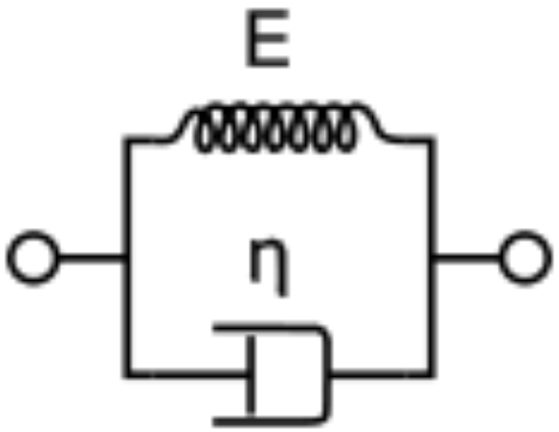
Hooke



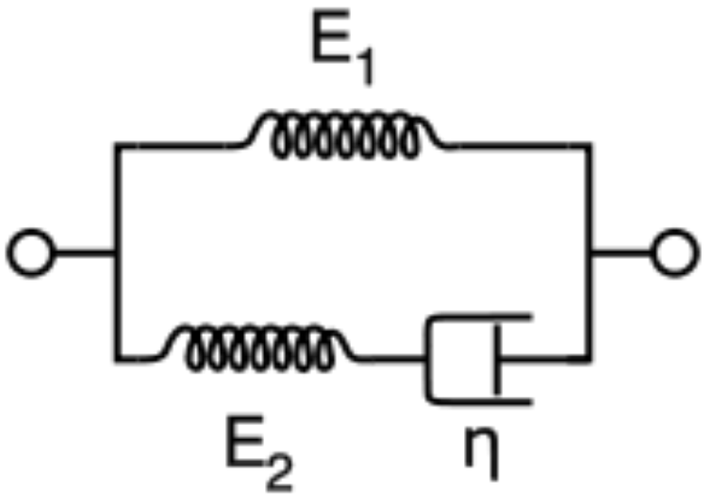
Newton



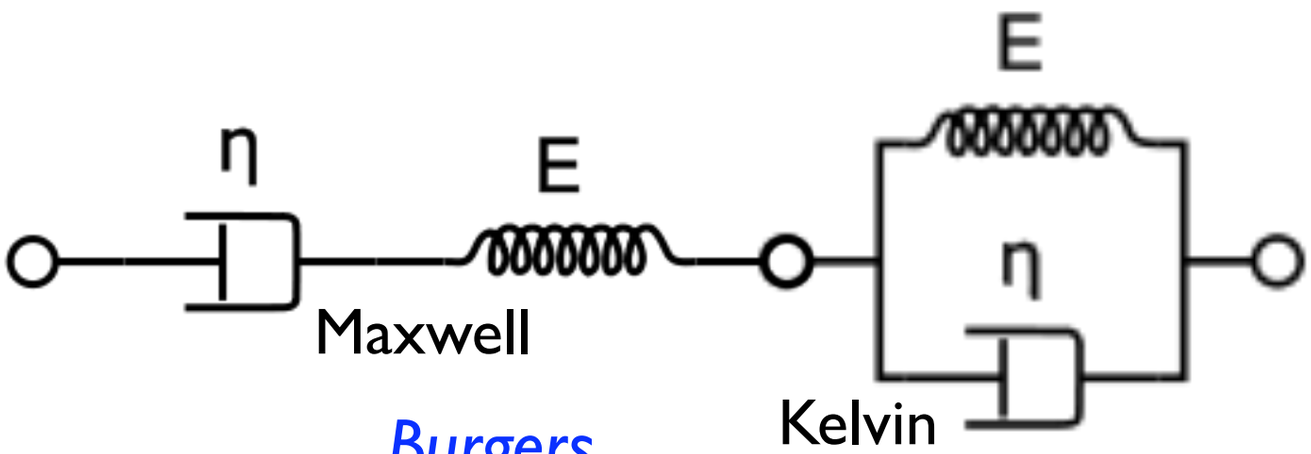
Maxwell



Kelvin-Voigt



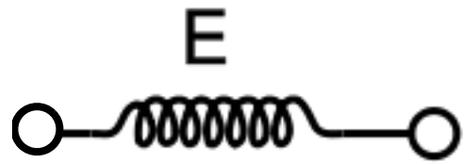
Standard Linear Solid



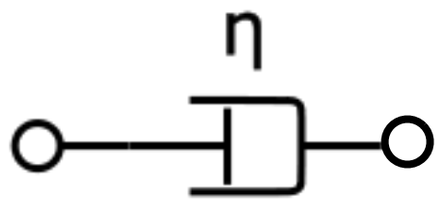
Burgers

Linear viscoelastic rheological models

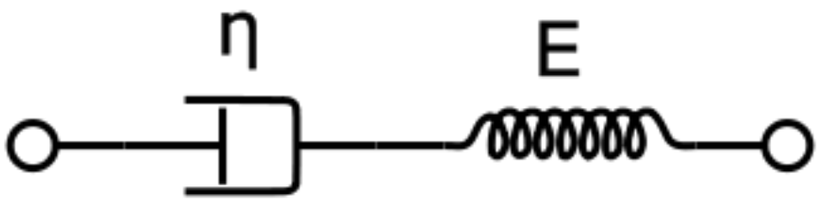
Seismology



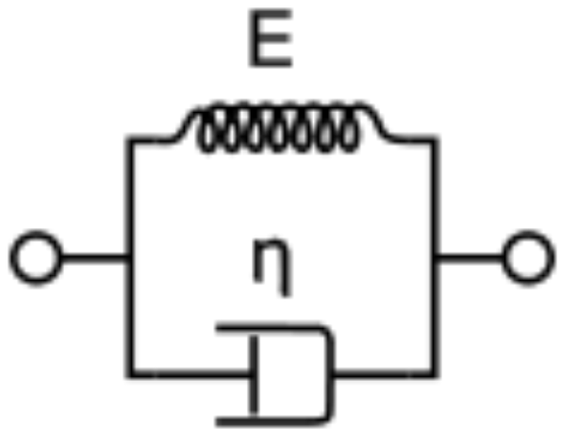
Hooke



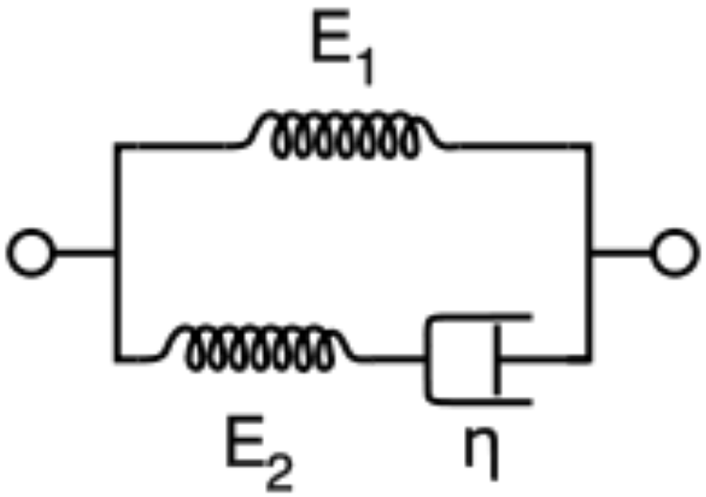
Newton



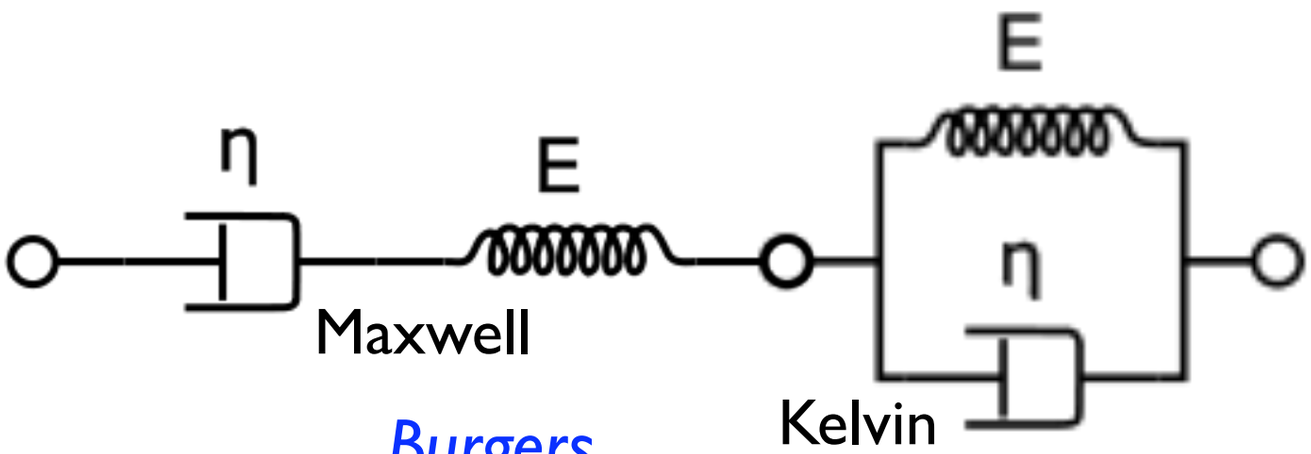
Maxwell



Kelvin-Voigt



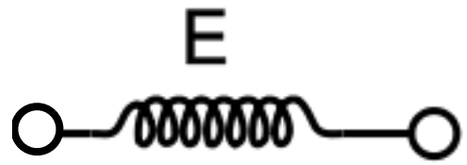
Standard Linear Solid



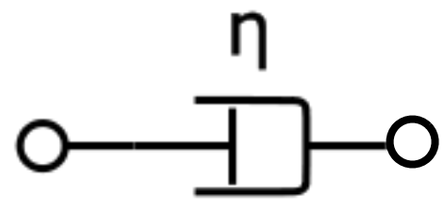
Burgers

Linear viscoelastic rheological models

Seismology

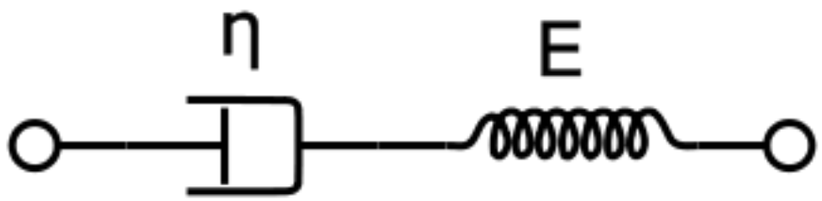


Hooke

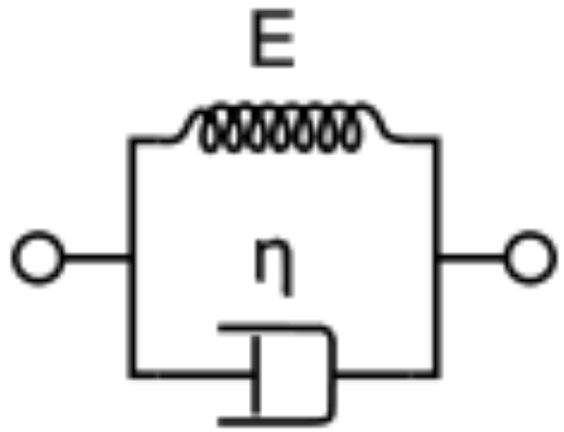


Newton

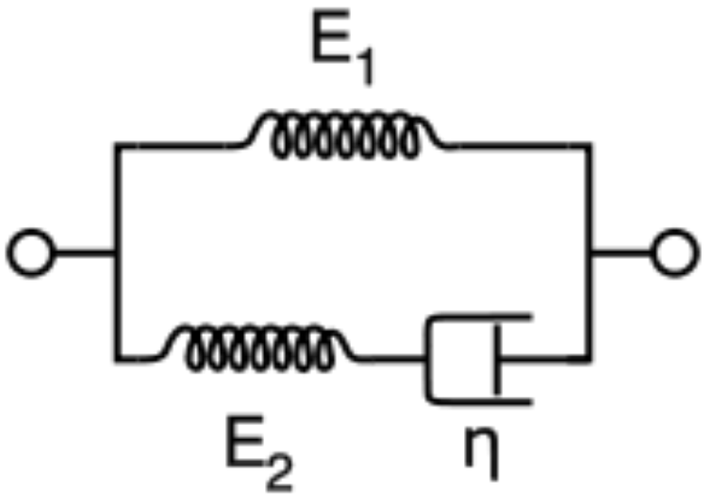
Mantle convection



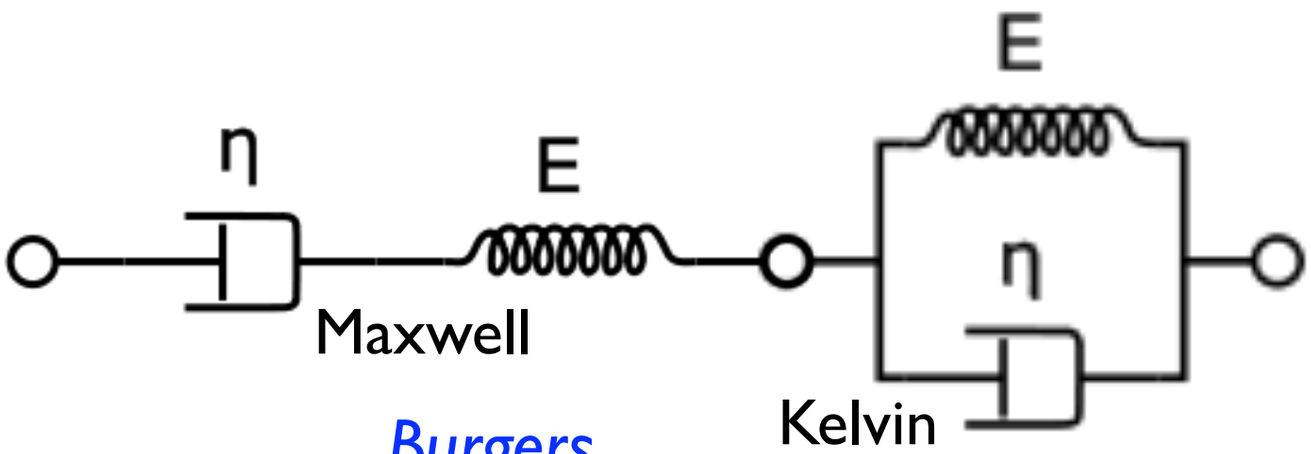
Maxwell



Kelvin-Voigt



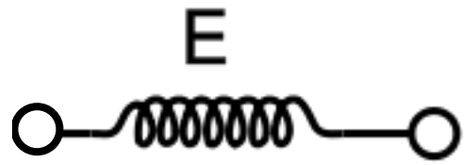
Standard Linear Solid



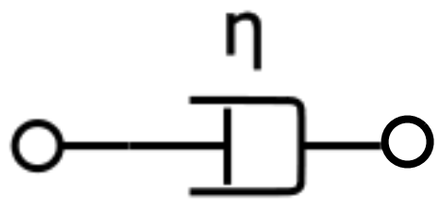
Burgers

Linear viscoelastic rheological models

Seismology



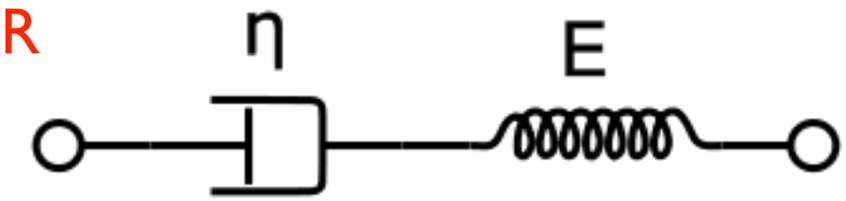
Hooke



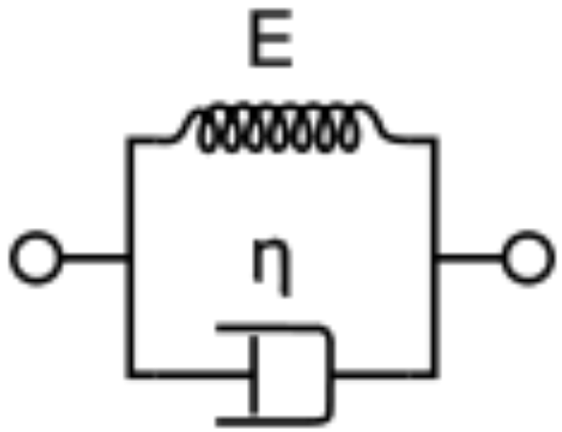
Newton

Mantle convection

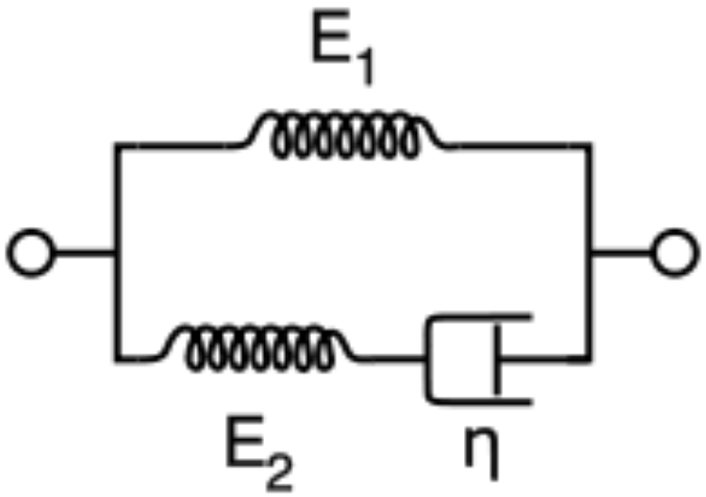
PGR



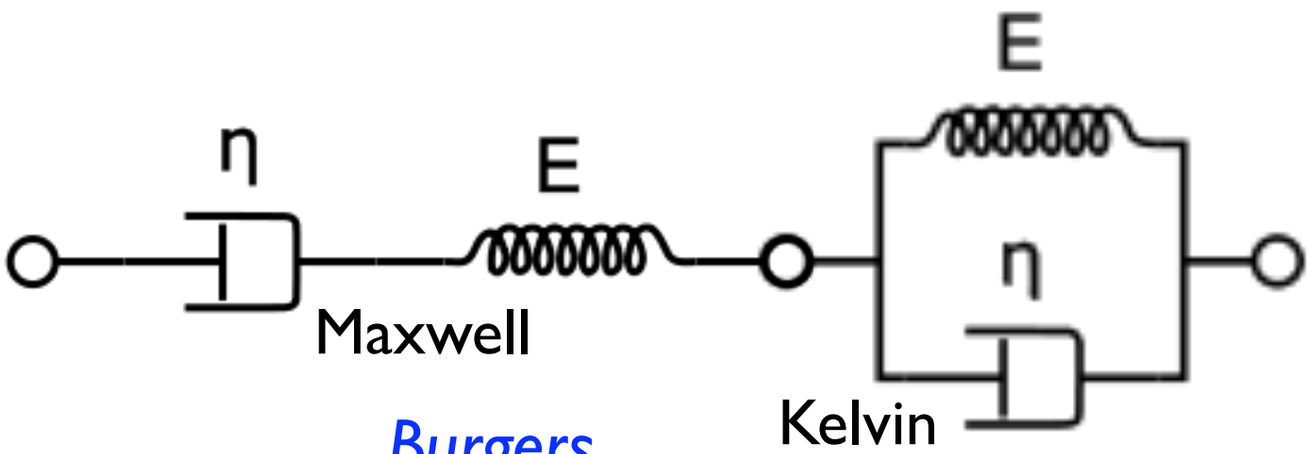
Maxwell



Kelvin-Voigt



Standard Linear Solid



Burgers

- Normal modes are sons of the "Laplace transform formula":

$$\tilde{x}(s) = \int_{0^-}^{\infty} e^{-st} x(t) dt \qquad \textit{Laplace transform}$$

$$x(t) = \frac{1}{2\pi i} \int_{Br} e^{st} \tilde{x}(s) ds \qquad \textit{Inverse transform}$$

But while inverse Laplace inverse transform ***is*** unique, the inversion formula **is not**.

Other formulas exist. One of these, the POST-WIDDER formula, is particularly attracting.

(*) See work of B. Vermeersen and others for all the other details.

- "Real" Laplace inversion formula of Post-Widder (~1930):

- Theorem:

$$x(t) = \lim_{n \mapsto \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \tilde{x}^{(n)} \left(\frac{n}{t}\right)$$

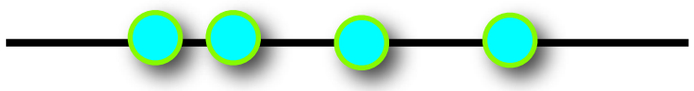
- For a rigorous demonstration of this theorem: see the Kurt Brian paper linked to wikipedia:

http://en.wikipedia.org/wiki/Post%27s_inversion_formula

- Indeed, other "real" (but probably less useful) formulas exist. See the references in Spada & Boschi GJI 2006

- "Real" Laplace inversion formula of Post-Widder (~1930):

$$x(t) = \lim_{n \mapsto \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \tilde{x}^{(n)}\left(\frac{n}{t}\right)$$

Usual method (NM)	*Post-Widder (PW)*
	
Re(s) < 0	Re(s) > 0
(singularities: poles)	(singularity-free)

- ADVANTAGES of PW formula:

$$x(t) = \lim_{n \mapsto \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \tilde{x}^{(n)}\left(\frac{n}{t}\right)$$

- simple looking,

- for Love numbers applications, allows to maintain the (elegant and analytical) propagators formalism,

- straightforward implementation of complex (linear) rheologies. One line of code for one rheology,

- root finder is not needed: the NMs simply vanish (they can be retrieved e. g., by least squares, if needed).

- DISADVANTAGES of PW formula:

$$x(t) = \lim_{n \mapsto \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \tilde{x}^{(n)} \left(\frac{n}{t}\right)$$

- it oscillates, and it has an extremely slow, logarithmic convergence *(but it can be efficiently accelerated),*

- a multi-precision environment is necessary (to avoid catastrophic cancellation) - *now available (and free),*

- not useful if the LT is singular on *Re(s)>0* (e. g., for compressible models) *(serious flaw),*

- Slow compared to Normal Modes,

- Difficult benchmarks with NM solutions, especially for exotic rheologies! *(this can be partly fixed).*

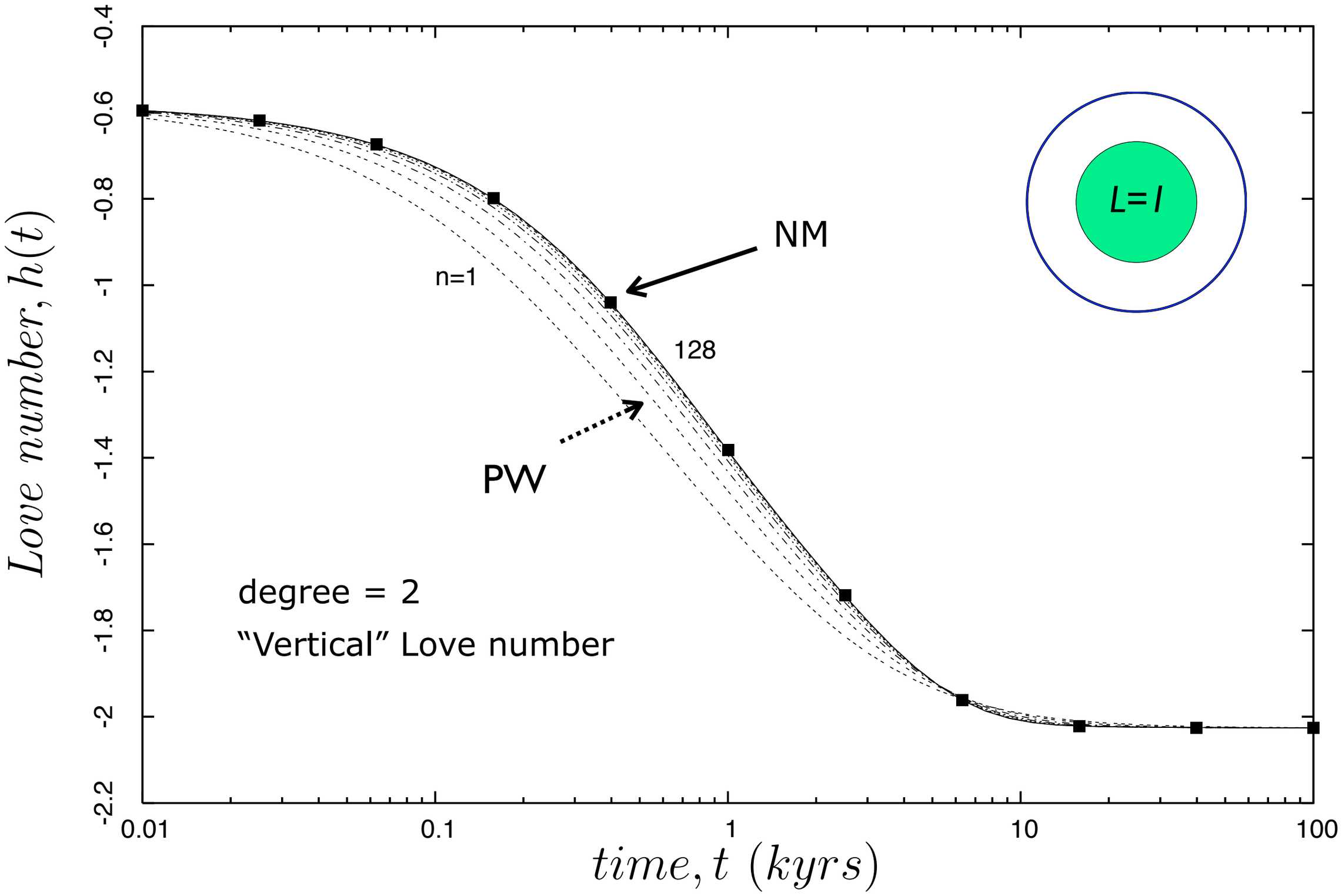
- Applications:

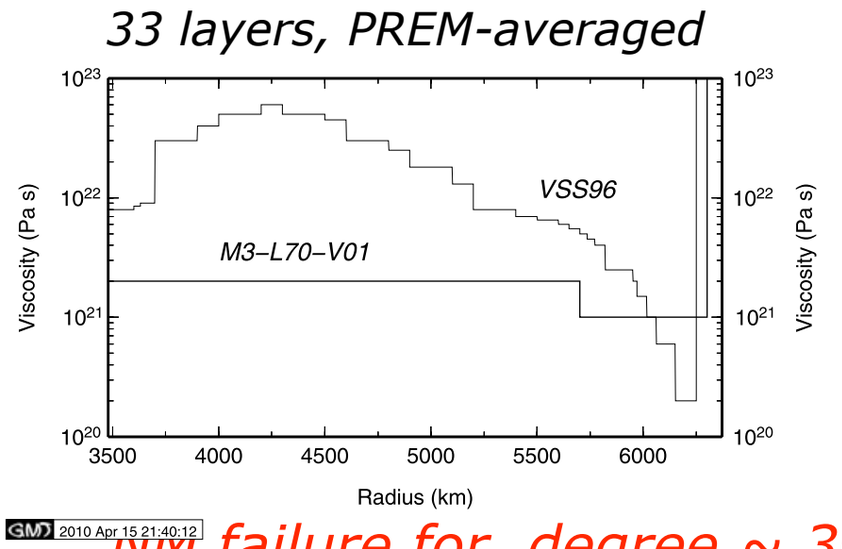
1) PW(G) vs. Normal Modes (NM),
2) Very low (Maxwell) viscosity layers,
3) Evidence of (numerical) failure of NM,
4) Generalized Maxwell (GM) rheology,
5) The case of “many” layers.

$$x(t) = \lim_{n \mapsto \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \tilde{x}^{(n)} \left(\frac{n}{t}\right)$$

Case study# 1

PW vs NM

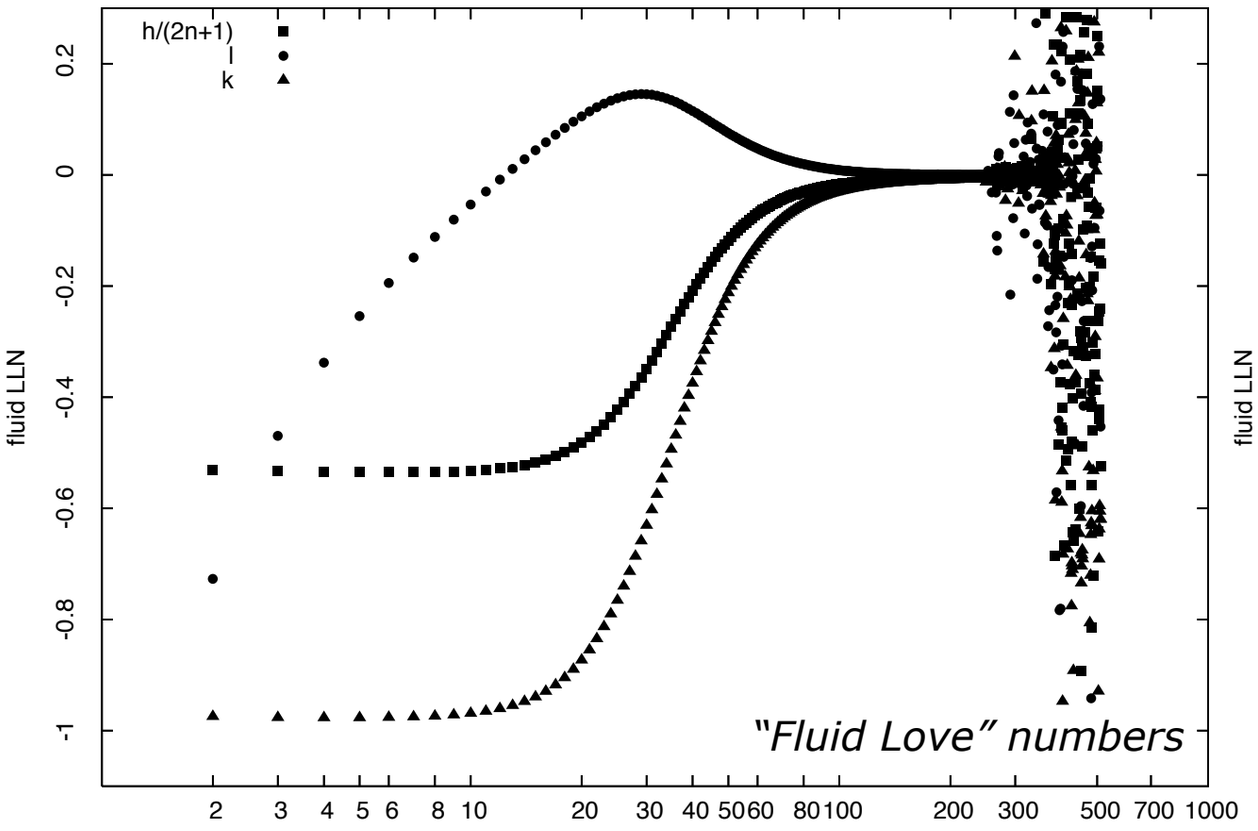


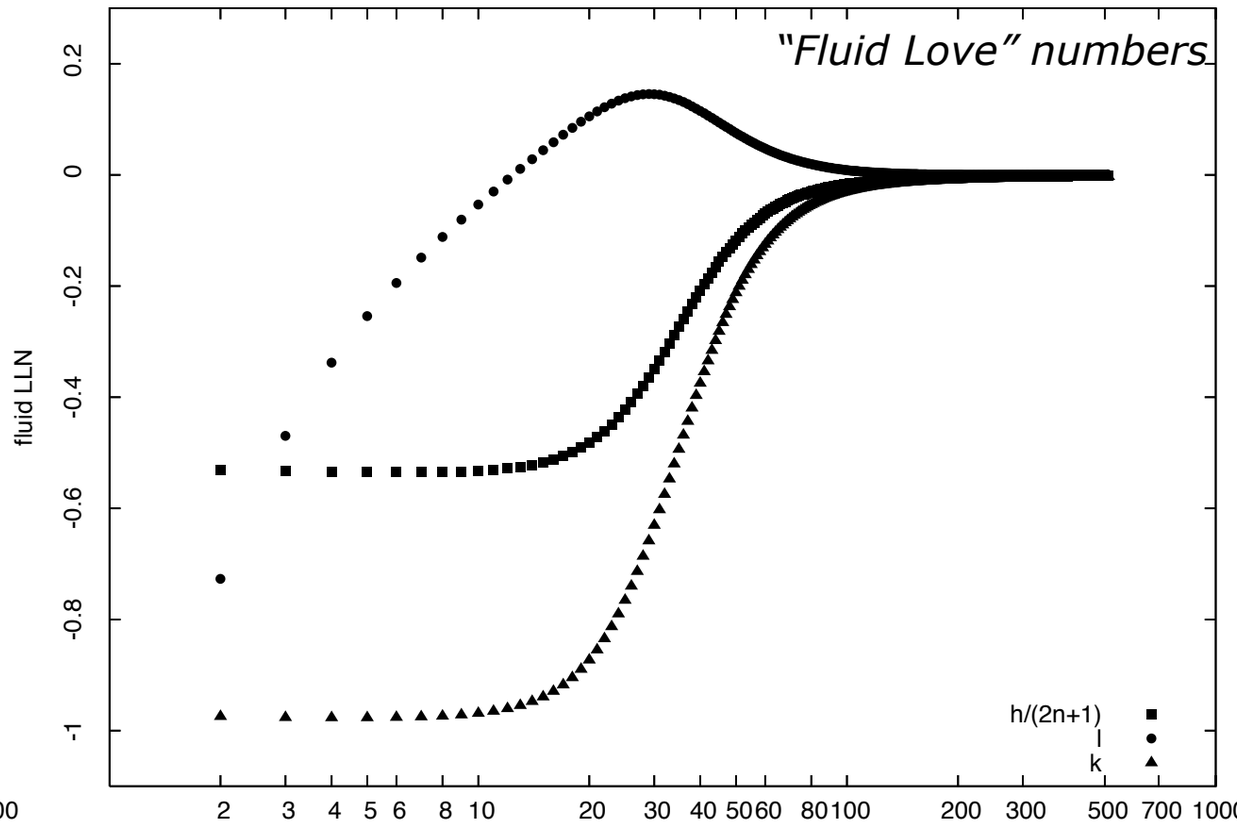


Case study# 2: a failure of NM

NM failure for degree ~ 300

PW response smooth to degree ~ 500

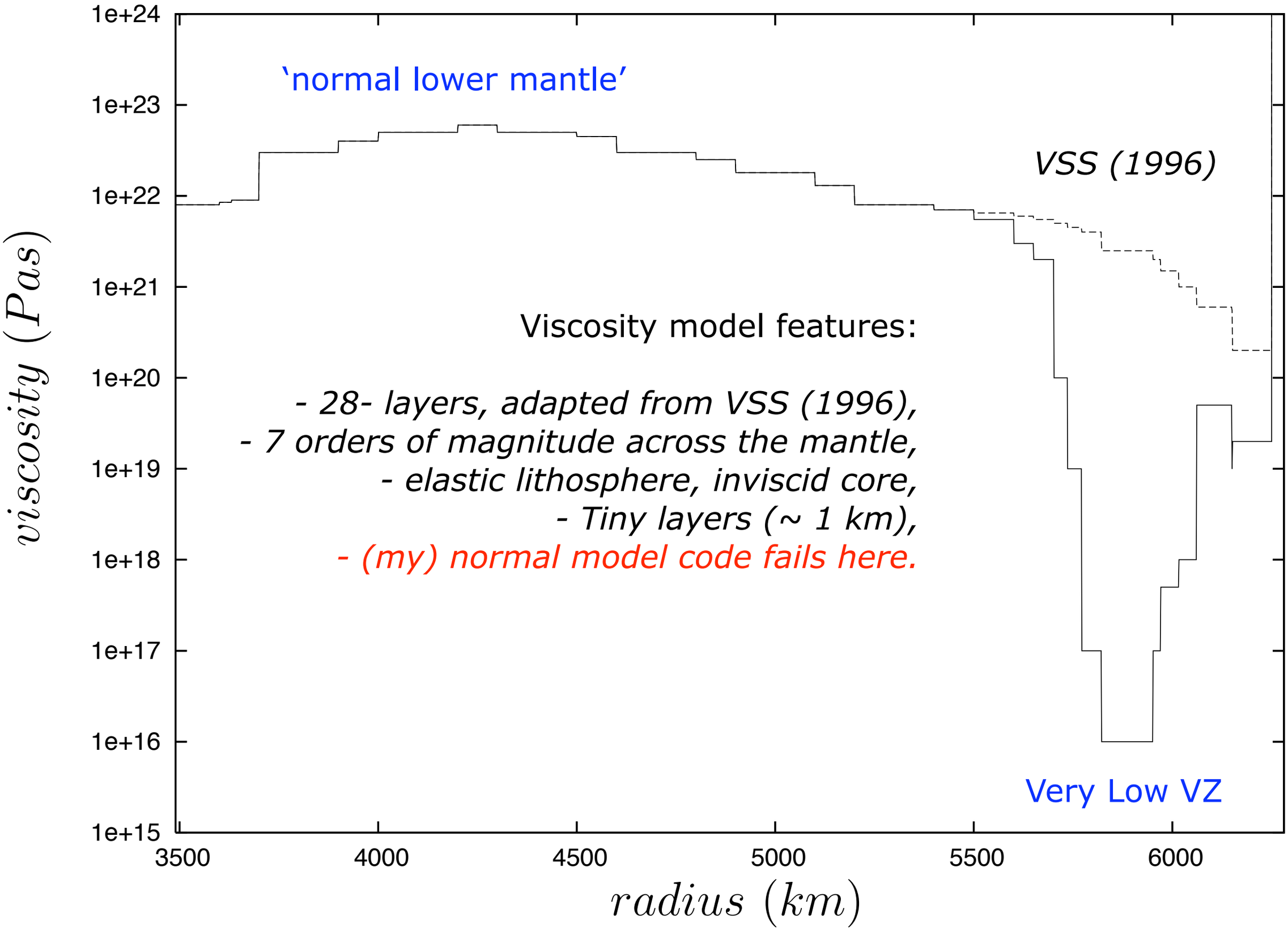




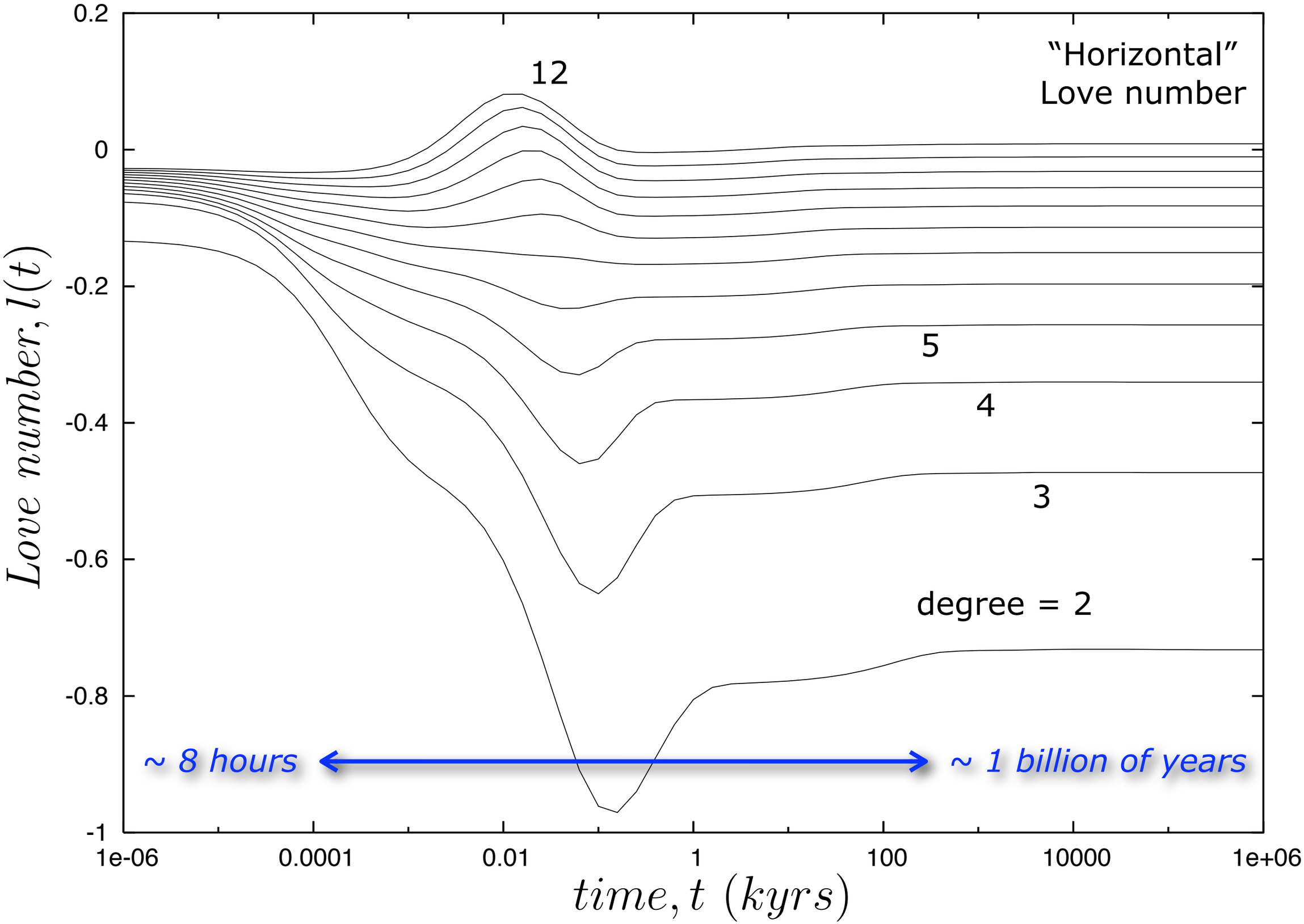
(Possible cause: root finder module in program TABOO)

not a proof of "exactness" of course, but a strong indication

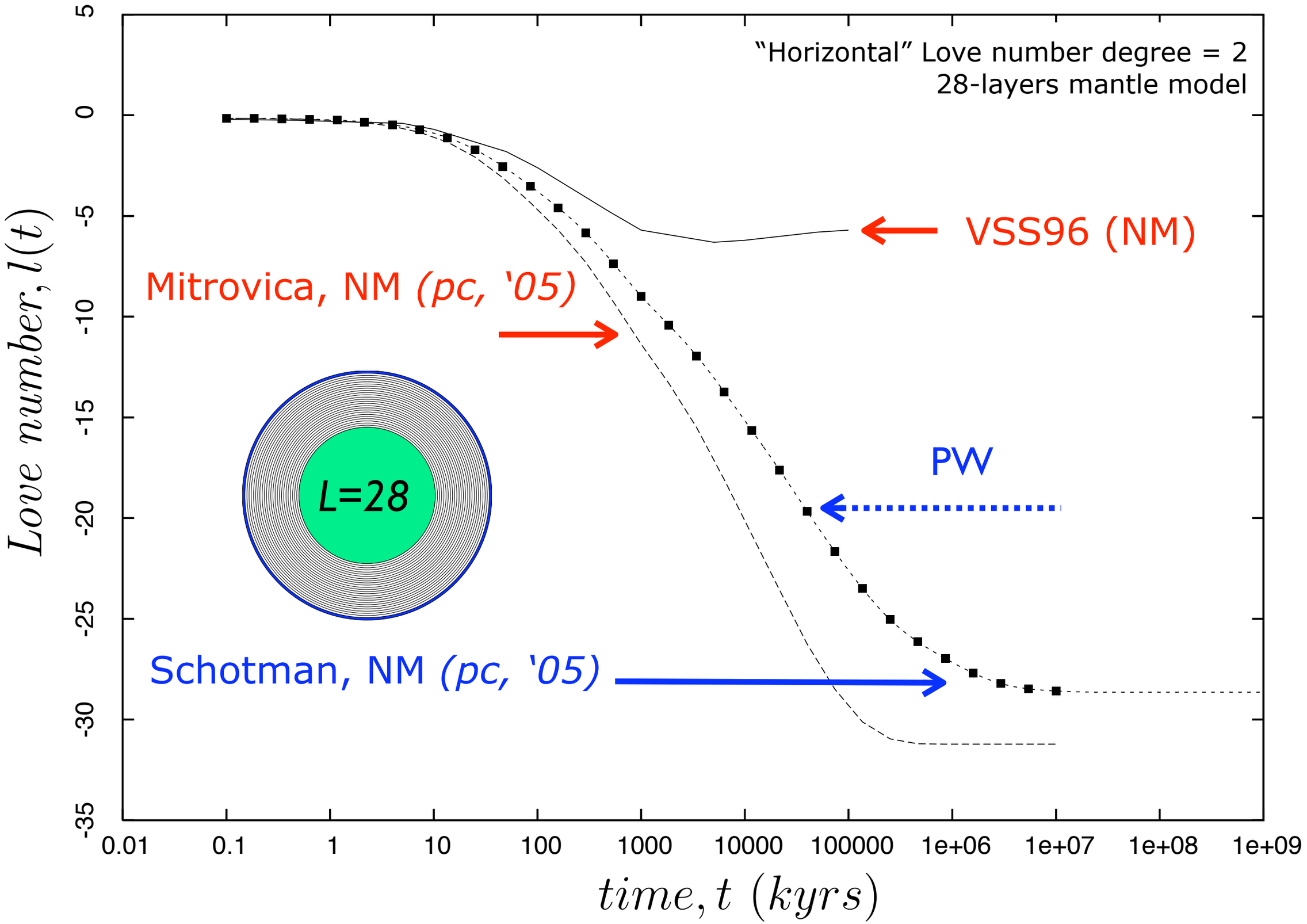
Case study# 2 Very low (Maxwell) viscosity layers



Case study# 2 Very low (Maxwell) viscosity layers

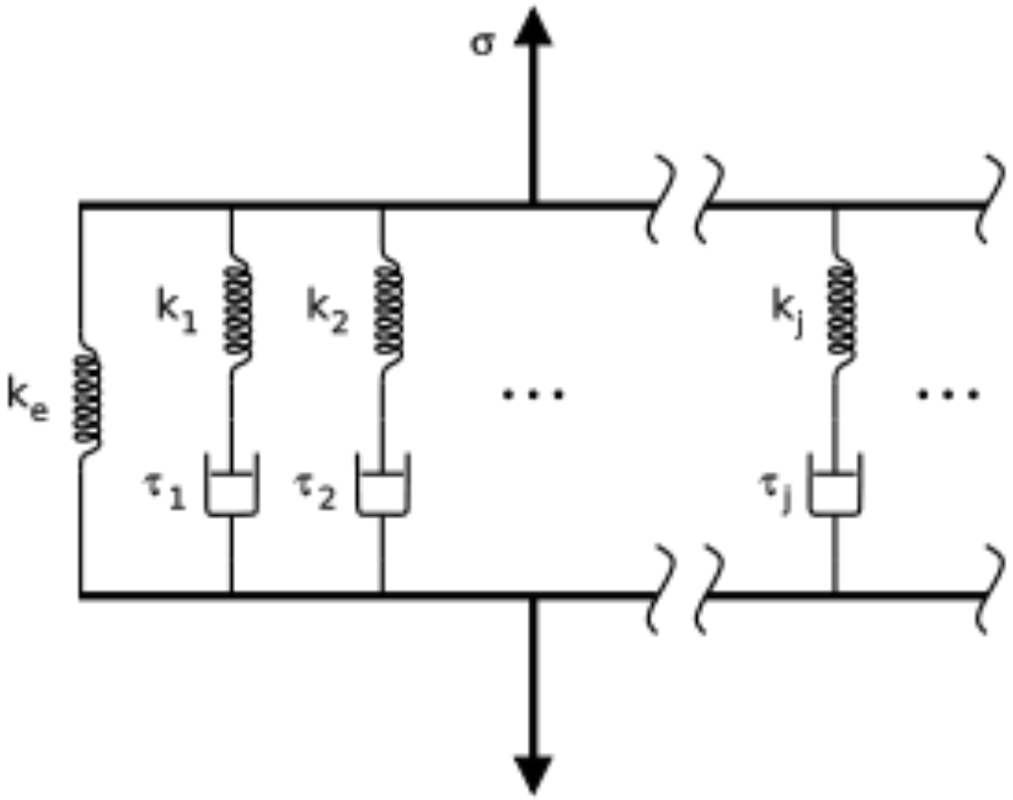


Case study# 3 failure of NM - see Spada & Boschi GJI 2006



Case study# 4

Generalized Maxwell (GM) rheology



Effective shear modulus

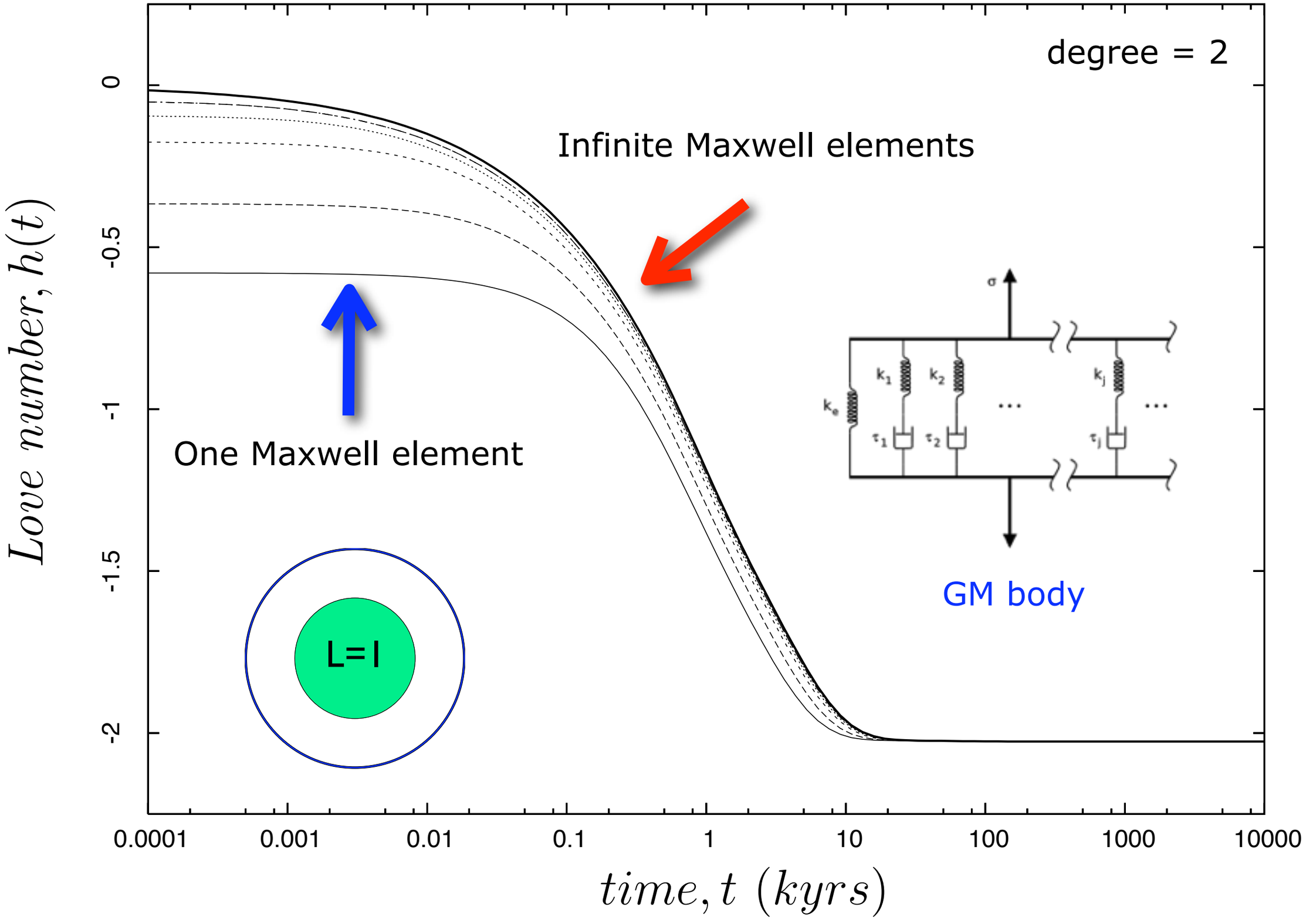
$$\tilde{\mu} = \sum_{k=1}^{\infty} \frac{\mu_k s}{s + \mu_k/\eta_k}$$

E.g., for $\mu_k = \mu_0/k^2$ and $\eta_k = \eta_0/k^2$ the series converge:

$$2\tilde{\mu}^{\infty} = \mu_0 \left[(\pi\sqrt{s\tau}) \coth(\pi\sqrt{s\tau}) - 1 \right], \quad \tau = \eta_0/\mu_0.$$

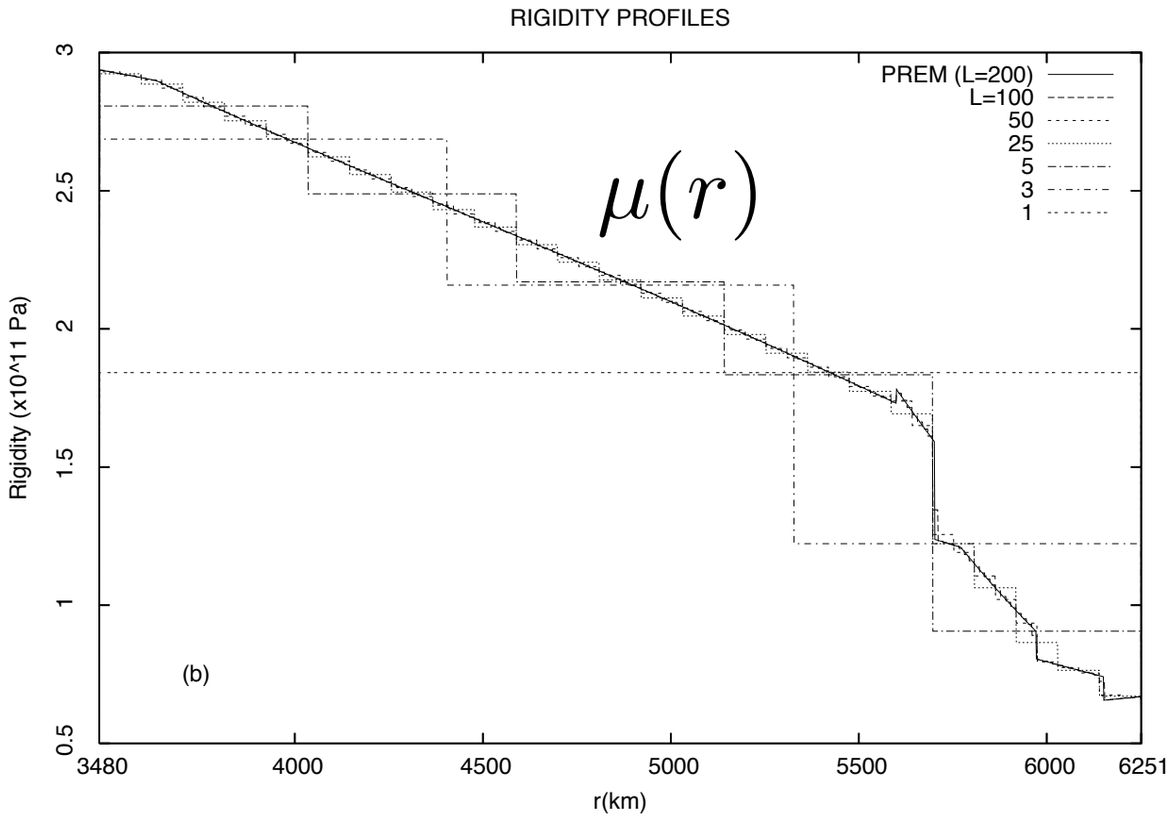
see Spada, Comput. Geosci. 2008

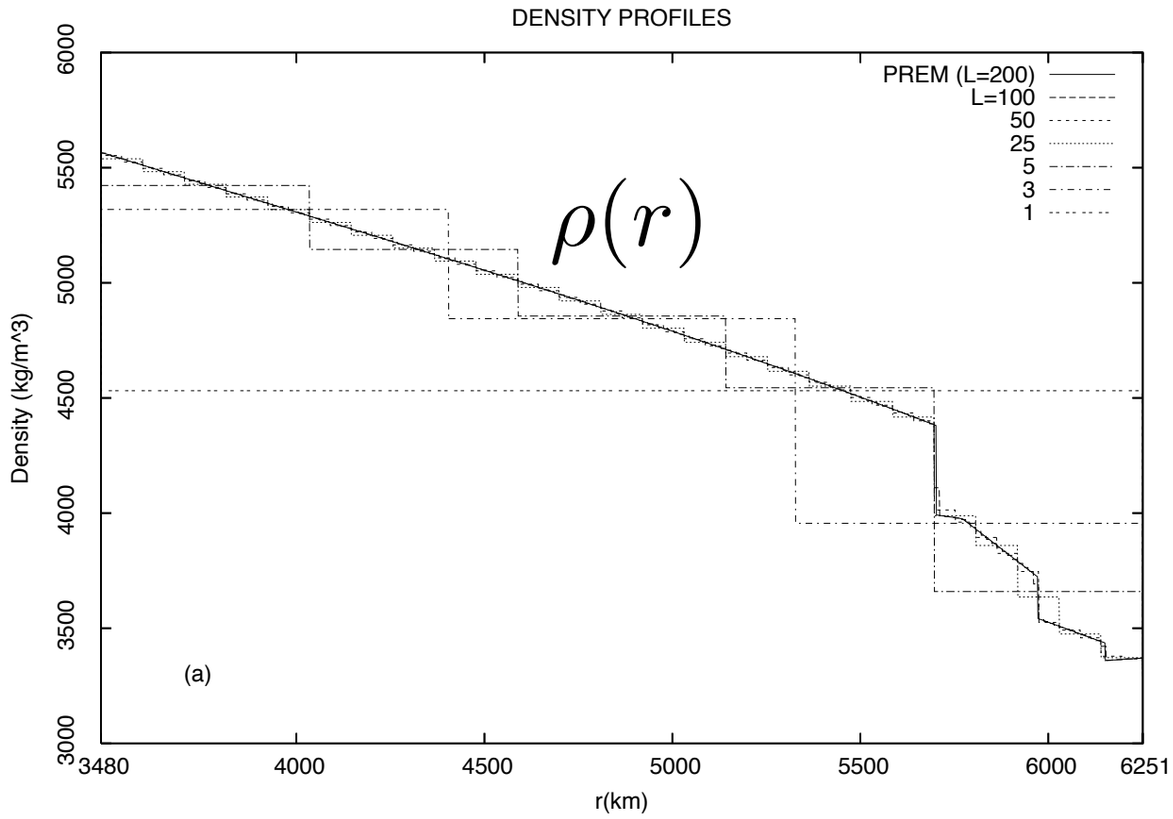
Case study# 4 an extremely complex rheology (MW)

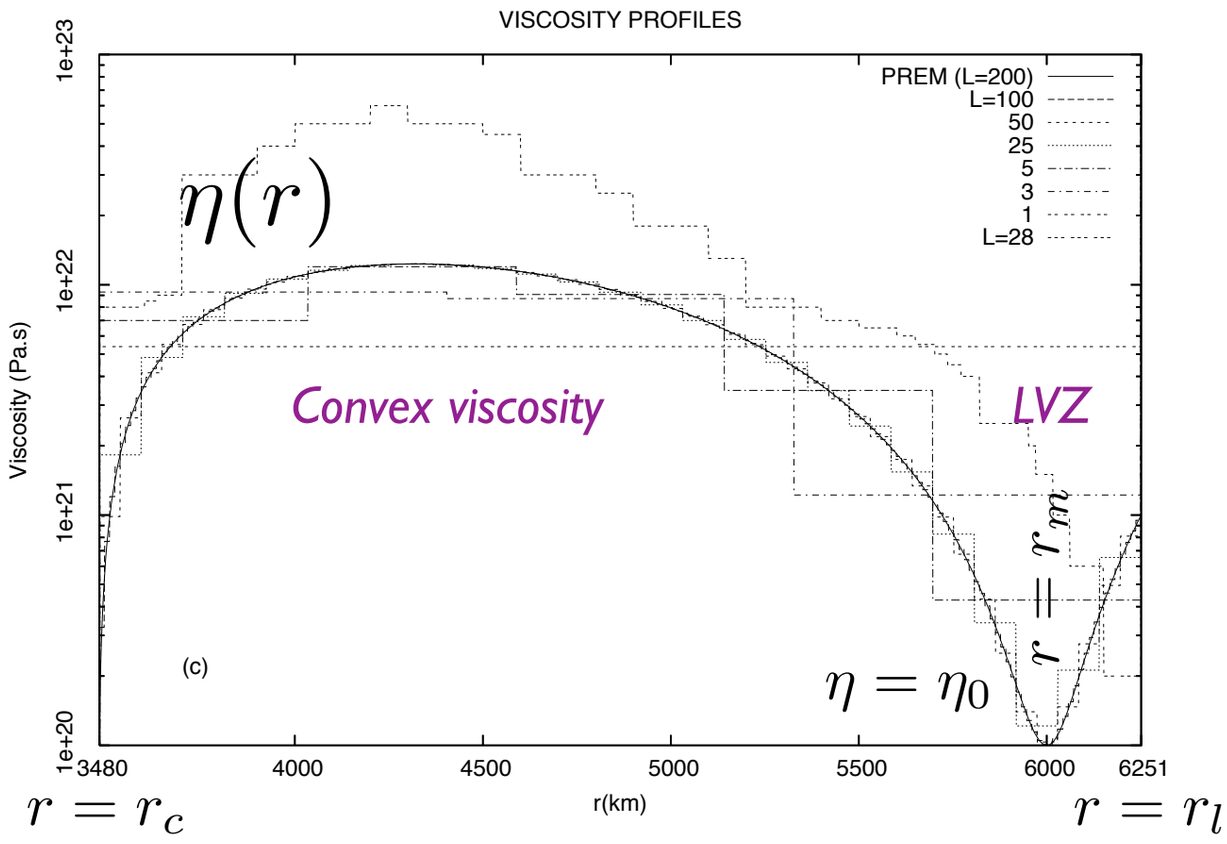


Case study# 5

The case of "many" layers (L=200)





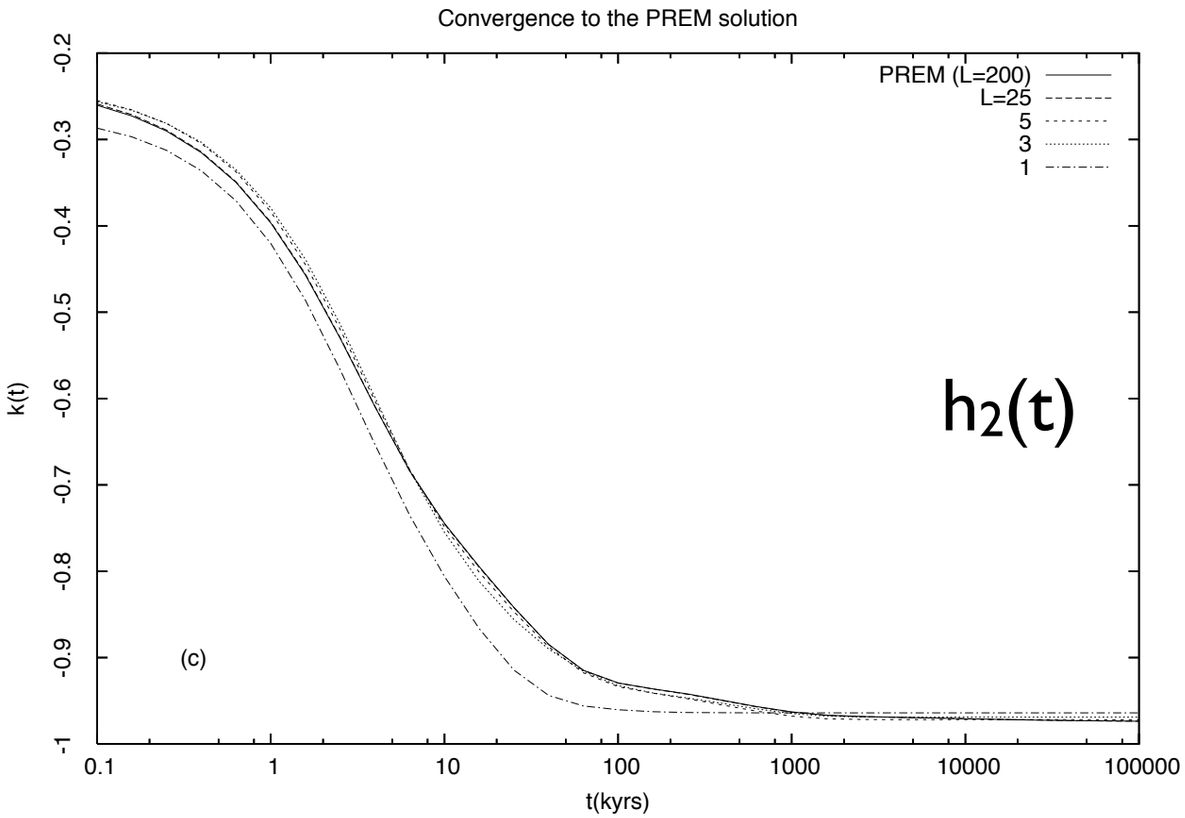


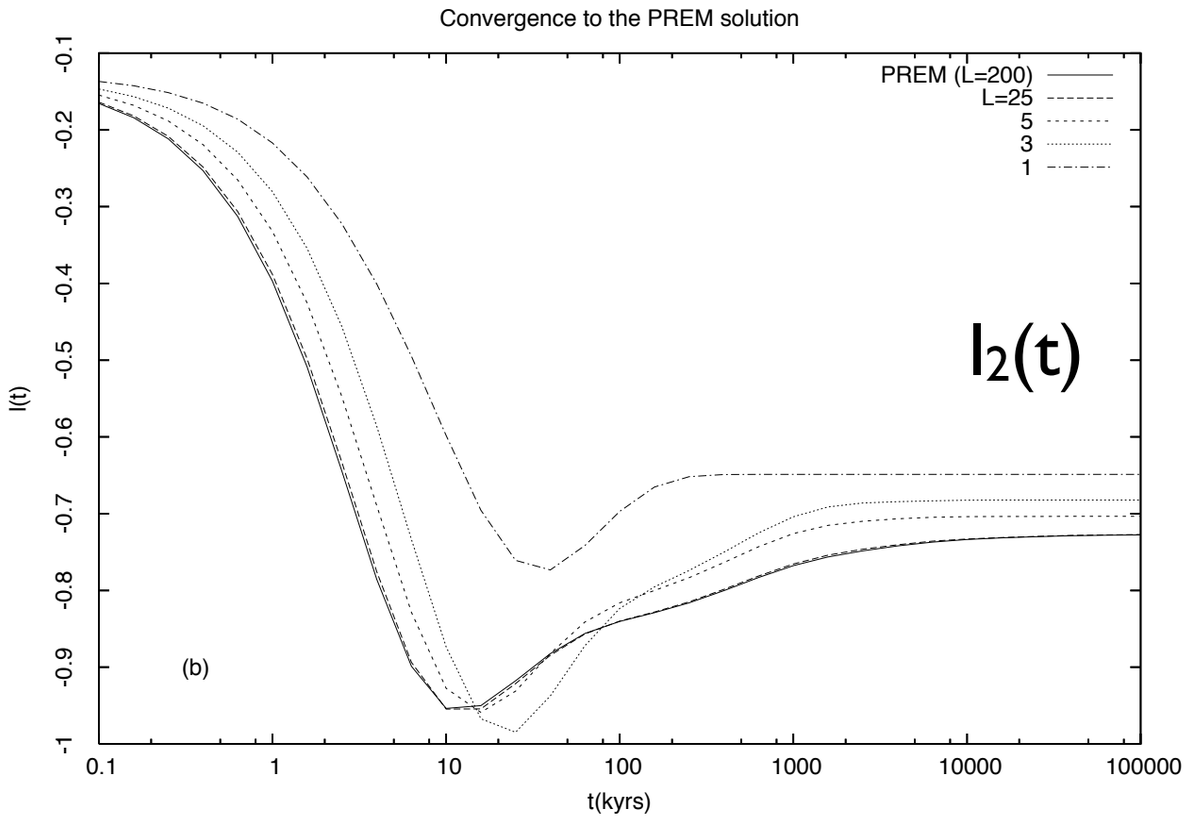
"Convex viscosity profile with a *Low Viscosity Zone"*

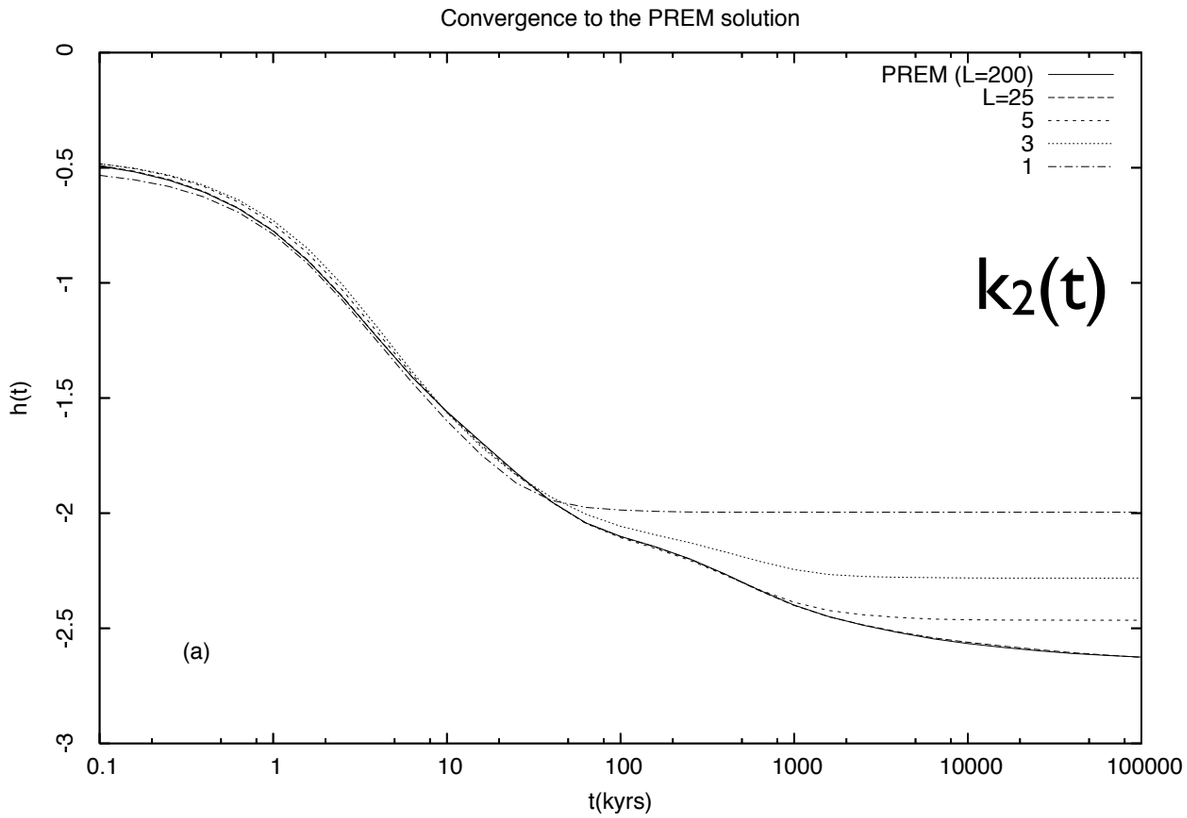
$$\frac{\eta(r)}{\eta_0} = \left[1 + \left(\frac{\eta_1 - \eta_0}{\eta_0}\right)\left(\frac{r - r_c}{r_l - r_c}\right)\left(\frac{r - r_m}{r_l - r_m}\right)^2\right]$$

Case study# 5

The case of "many" layers (L=200)







- "Exhumation" of the PW formula:

  - *PW: Theory and a source-free code:*

Spada, G., Boschi, L. 2006. Using the Post-Widder formula to compute the Earth's viscoelastic Love numbers, Geophys. J. Int., 166 (1), 309-321.

Spada, G., 2008. ALMA, a Fortran program for computing the visco-elastic Love numbers of a spherically symmetric planet, Comput. and Geosci., 34 (6), 667-687.

  - *NM: Theory and a source-free code:*

http://www.samizdat.mines.edu/spada

Possible applications:

✓Post-glacial deformations (surface loads),
✓Glacio-isostatic adjustment and sea level problem,
✓Post-seismic deformations & stress diffusion,
✓Poloidal/Toroidal deformations,
✓Earth rotational variations (PM, LOD)
✓Mantle dynamics (internal loads)...