

Anomalous Polarization

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Can one and the same slowness surface have
different polarization?

A recent reminder that not all is as simple as one believes

A theorem by Bucataru, Boná and Slawinski (July 2004)

The symmetry of the slowness surface is the symmetry of the medium

is disproved by Bucataru's counter example (sent in an e-mail late September 2004):

$$C = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & -2 & 0 & 0 & 0 \\ 1 & -2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Because:

The medium is stable and orthotropic, but

the slowness surface is cubic! The characteristic equation is

$$\begin{aligned} -\lambda^3 + 6(x^2 + y^2 + z^2)\lambda^2 - \left(9(x^4 + y^4 + z^4) + 24.75(y^2z^2 + x^2z^2 + y^2x^2)\right)\lambda \\ + 18.75(x^2 + y^2 + z^2)(x^2y^2 + y^2z^2 + z^2x^2) + 57.75x^2y^2z^2 = 0 \end{aligned}$$

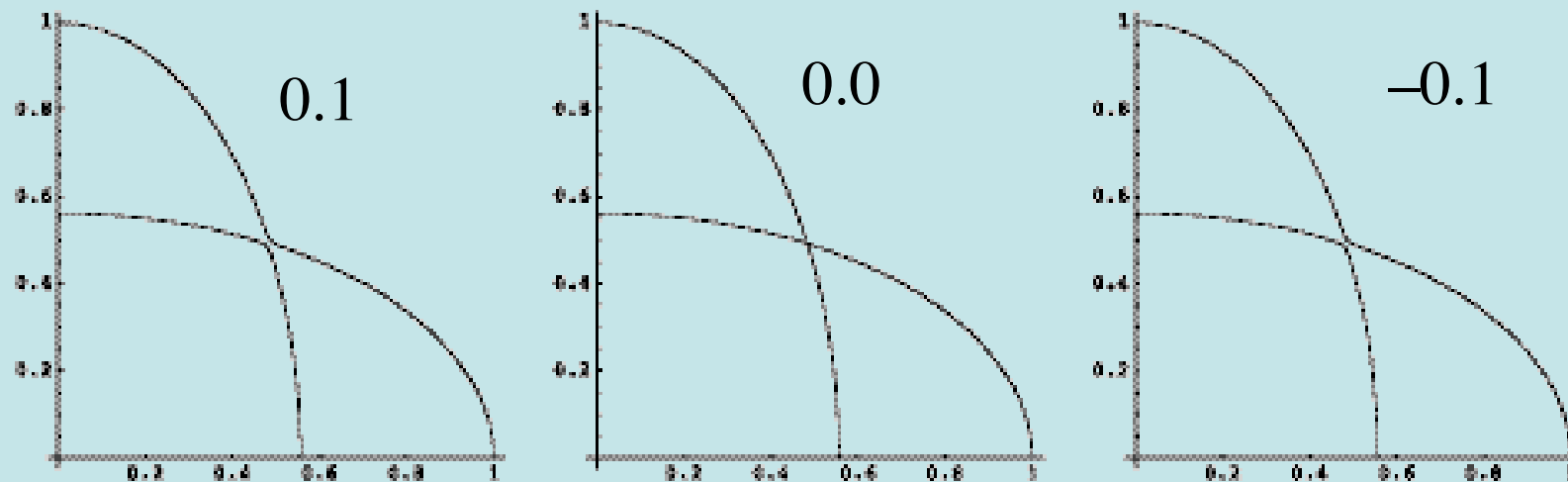
i.e. totally symmetric in x, y, z .

What is wrong??

Another story:

Long ago, when we still worked on TI, I visited a colleague who tried to make P- and SV-sheets of the slowness surface intersect by juggling the “shape-parameter” ($C_{13}+C_{55}$).

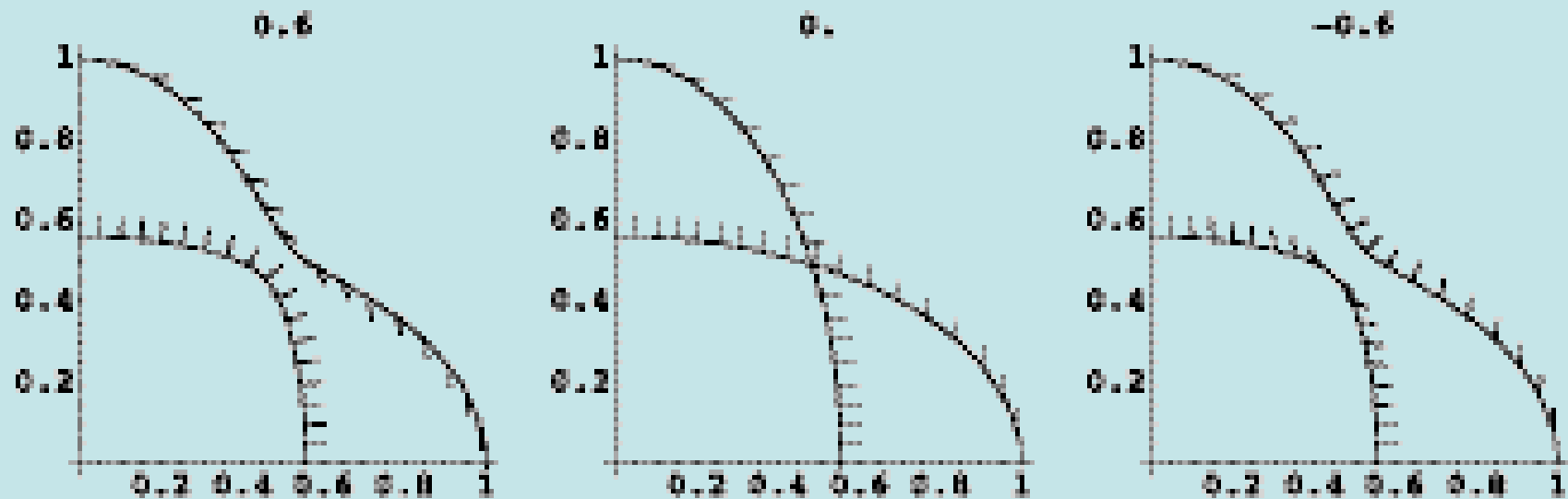
Whatever he tried, the best was a sequence like this:



So again, what is wrong??

I suggested to plot the polarization vectors:

And indeed:



The two outer panels are “**anomalous polarization companions**”. They have identical slowness surfaces, but the polarization vector on the l.h.s. rotates (in general) in the same direction as the propagation vector, that on the r.h.s. in the opposite direction.

The central panel shows the intersection of two ellipses with spatially fixed polarization. Note that it exists only for $C_{13}+C_{55}$ *exactly* = 0, and therefore has no practical significance.

General Question: Does anything defined by equalities have a practical significance?

General conditions for anomalous polarization

Anomalous polarization exists if a Christoffel matrix Γ and a distinct “anomalous companion” matrix Γ^ have the same characteristic equation, i.e., if $\det(\Gamma^* - I\lambda) = \det(\Gamma - I\lambda)$.*

$$\det(\Gamma - I\lambda) = -\lambda^3 + (\Gamma_{11} + \Gamma_{22} + \Gamma_{33})\lambda^2 - (\Gamma_{22}\Gamma_{33} - \Gamma_{23}^2 + \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2 + \Gamma_{11}\Gamma_{22} - \Gamma_{12}^2)\lambda + \Gamma_{11}\Gamma_{22}\Gamma_{33} + 2\Gamma_{23}\Gamma_{13}\Gamma_{12} - \Gamma_{11}\Gamma_{23}^2 - \Gamma_{22}\Gamma_{13}^2 - \Gamma_{33}\Gamma_{12}^2$$

Thus two stiffness tensors have identical characteristic equations (i.e., the same slowness surface) if and only if for all directions

- (i) *the diagonal terms* of the Christoffel matrices Γ and Γ^* ,
 - (ii) *the squares of the off-diagonal terms*, and
 - (iii) *the product of the three off-diagonal terms*
- are identical

The diagonal terms of Γ

$$\Gamma_{11} = c_{1j1l} n_j n_l = C_{11} n_1^2 + C_{66} n_2^2 + C_{55} n_3^2 + 2(C_{16} n_1 n_2 + C_{15} n_1 n_3 + C_{56} n_2 n_3) ,$$

$$\Gamma_{22} = c_{2j2l} n_j n_l = C_{66} n_1^2 + C_{22} n_2^2 + C_{44} n_3^2 + 2(C_{26} n_1 n_2 + C_{46} n_1 n_3 + C_{24} n_2 n_3) ,$$

$$\Gamma_{33} = c_{3j3l} n_j n_l = C_{55} n_1^2 + C_{44} n_2^2 + C_{33} n_3^2 + 2(C_{45} n_1 n_2 + C_{35} n_1 n_3 + C_{34} n_2 n_3) .$$

The two stiffness matrices must share 15 stiffnesses, namely

$$C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66}, C_{15}, C_{16}, C_{24}, C_{26}, C_{34}, C_{35}, C_{45}, C_{46}, \text{ and } C_{56} .$$

They can differ only in 6 stiffnesses, namely

$$C_{23}, C_{13}, C_{12}, C_{14}, C_{25}, \text{ and } C_{36} ,$$

The “free” stiffnesses

Anomalous companion matrices can differ only in

$C_{23}, C_{13}, C_{12}, C_{14}, C_{25}$, and C_{36} ,

C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}
	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
		C_{33}	C_{34}	C_{35}	C_{36}
			C_{44}	C_{45}	C_{46}
				C_{55}	C_{56}
					C_{66}

The off-diagonal terms of Γ

$$\begin{aligned}\Gamma_{23} &= (C_{23} + C_{44})n_2n_3 + (C_{36} + C_{45})n_1n_3 + (C_{25} + C_{46})n_1n_2 + C_{56}n_1^2 + C_{24}n_2^2 + C_{34}n_3^2, \\ \Gamma_{13} &= (C_{36} + C_{45})n_2n_3 + (C_{13} + C_{55})n_1n_3 + (C_{14} + C_{56})n_1n_2 + C_{15}n_1^2 + C_{46}n_2^2 + C_{35}n_3^2, \\ \Gamma_{12} &= (C_{25} + C_{46})n_2n_3 + (C_{14} + C_{56})n_1n_3 + (C_{12} + C_{66})n_1n_2 + C_{16}n_1^2 + C_{26}n_2^2 + C_{45}n_3^2.\end{aligned}$$

Since only the stiffnesses in the white fields of the Figure may be different in anomalous companion matrices, the nine stiffnesses C_{15} , C_{16} , C_{24} , C_{26} , C_{34} , C_{35} , C_{45} , C_{46} , and C_{56} – i.e., the factors of the squares of components of the propagation vector – must vanish, and the off-diagonal terms of the Christoffel matrix in a pair of companion matrices must have the form

The off-diagonal terms of Γ and Γ^*

$$\Gamma_{23} = (C_{23} + C_{44})n_2n_3 + C_{36}n_1n_3 + C_{25}n_1n_2 ,$$

$$\Gamma_{13} = C_{36}n_2n_3 + (C_{13} + C_{55})n_1n_3 + C_{14}n_1n_2 ,$$

$$\Gamma_{12} = C_{25}n_2n_3 + C_{14}n_1n_3 + (C_{12} + C_{66})n_1n_2$$

and

$$\Gamma_{23}^* = (C_{23}^* + C_{44})n_2n_3 + C_{36}^*n_1n_3 + C_{25}^*n_1n_2 ,$$

$$\Gamma_{13}^* = C_{36}^*n_2n_3 + (C_{13}^* + C_{55})n_1n_3 + C_{14}^*n_1n_2 ,$$

$$\Gamma_{12}^* = C_{25}^*n_2n_3 + C_{14}^*n_1n_3 + (C_{12}^* + C_{66})n_1n_2 ,$$

with

$$\Gamma_{23}^* = \pm\Gamma_{23}, \quad \Gamma_{13}^* = \pm\Gamma_{13}, \quad \text{and} \quad \Gamma_{12}^* = \pm\Gamma_{12} .$$

The last line allows eight sign combinations:

Sign combinations

Γ_{23}	+	+	−	−	−	−	+	+
Γ_{13}	+	−	+	−	−	+	−	+
Γ_{12}	+	−	−	+	−	+	+	−
$\Gamma_{23} \Gamma_{13} \Gamma_{12}$	+	+	+	+	−	−	−	−

Four sign combination *maintain* the sign of the product $\Gamma_{23} \Gamma_{13} \Gamma_{12}$,
four sign combination *invert* the sign of the product $\Gamma_{23} \Gamma_{13} \Gamma_{12}$. ■

The four members of either set have identical slowness surfaces

The slowness surfaces of the two sets are distinct, but have the same curves of intersection with the coordinate planes.

Only one of the three “upper-right” stiffnesses may change

In a companion pair, *precisely two* of the off-diagonal terms differ in sign. Inspection of

$$\Gamma_{23} = (C_{23} + C_{44})n_2n_3 + C_{36}n_1n_3 + C_{25}n_1n_2 ,$$

$$\Gamma_{13} = C_{36}n_2n_3 + (C_{13} + C_{55})n_1n_3 + C_{14}n_1n_2 ,$$

$$\Gamma_{12} = C_{25}n_2n_3 + C_{14}n_1n_3 + (C_{12} + C_{66})n_1n_2$$

shows that this happens if simultaneously one of the following holds:

- (i) $C_{23}+C_{44}$, $C_{13}+C_{55}$, and C_{36} change sign and C_{14} , C_{25} vanish,
- (ii) $C_{13}+C_{55}$, $C_{12}+C_{66}$, and C_{14} change sign and C_{36} , C_{25} vanish,
- (iii) $C_{12}+C_{66}$, $C_{23}+C_{44}$, and C_{25} change sign and C_{14} , C_{36} vanish.

It follows that any medium with an anomalous companion is at least of monoclinic symmetry

Three monoclinic media with anomalous companions

$$\begin{pmatrix} C_{11} & C_{12} & \bullet & 0 & 0 & 0 \\ & C_{22} & \bullet & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & \circ \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix}, \begin{pmatrix} C_{11} & \bullet & \bullet & \circ & 0 & 0 \\ & C_{22} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & \circ & & & C_{66} \end{pmatrix}, \begin{pmatrix} C_{11} & \bullet & C_{13} & 0 & 0 & 0 \\ & C_{22} & \bullet & 0 & \circ & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix}$$

The “•” indicate stiffnesses which must change in magnitude and sign, the “◦” indicate stiffnesses that must change sign or vanish in order to change a medium into its anomalous companion medium.

The three media are equivalent under cyclic exchange of subscripts.

Stability

Formally one can always find an anomalous companion to a medium that satisfies the conditions laid down in the previous slides. Such a companion is meaningful *iff* both media are stable.

Stability means that *any* strain requires energy. For an arbitrary strain, e.g., $\{\varepsilon_2, \varepsilon_4, \varepsilon_5\}$, one has

$$2E = \varepsilon_{ij} \sigma_{ij} \Rightarrow \begin{pmatrix} 0 \\ \varepsilon_2 \\ 0 \\ \varepsilon_4 \\ \varepsilon_5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{34} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} 0 \\ \varepsilon_2 \\ 0 \\ \varepsilon_4 \\ \varepsilon_5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon_2 \\ 0 \\ \varepsilon_4 \\ \varepsilon_5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} C_{12}\varepsilon_2 + C_{14}\varepsilon_4 + C_{15}\varepsilon_5 \\ C_{22}\varepsilon_2 + C_{24}\varepsilon_4 + C_{25}\varepsilon_5 \\ C_{23}\varepsilon_2 + C_{34}\varepsilon_4 + C_{35}\varepsilon_5 \\ C_{24}\varepsilon_2 + C_{44}\varepsilon_4 + C_{45}\varepsilon_5 \\ C_{25}\varepsilon_2 + C_{45}\varepsilon_4 + C_{55}\varepsilon_5 \\ C_{26}\varepsilon_2 + C_{46}\varepsilon_4 + C_{56}\varepsilon_5 \end{pmatrix} =$$

$$C_{22}\varepsilon_2^2 + C_{44}\varepsilon_4^2 + C_{55}\varepsilon_5^2 + 2(C_{24}\varepsilon_2\varepsilon_4 + C_{25}\varepsilon_2\varepsilon_5 + C_{45}\varepsilon_4\varepsilon_5)$$

This “quadratic form” is positive for any values of $\{\varepsilon_2, \varepsilon_4, \varepsilon_5\}$ if the determinant

$$\begin{vmatrix} C_{22} & C_{24} & C_{25} \\ C_{24} & C_{44} & C_{45} \\ C_{25} & C_{45} & C_{55} \end{vmatrix} > 0 .$$

Stability 2

It follows from the previous slide that a medium is stable if the determinants of all “principal minors” of its stiffness matrix are > 0 .

This can be reduced to “the determinants of all *leading* principal minors”. For the monoclinic medium with $C_{14} \neq 0$ we have then

$$C_{11} > 0; C_{22} > 0; C_{33} > 0; C_{44} > 0; C_{55} > 0; C_{66} > 0;$$

$$-\sqrt{C_{22}C_{33}} < C_{23} < \sqrt{C_{22}C_{33}};$$

$$-\sqrt{C_{11}C_{33}} < C_{13} < \sqrt{C_{11}C_{33}};$$

$$-\sqrt{C_{11}C_{22}} < C_{12} < \sqrt{C_{11}C_{22}};$$

$$C_{11}C_{22}C_{33} + 2C_{23}C_{13}C_{12} - C_{23}^2C_{11} - C_{13}^2C_{22} - C_{12}^2C_{33} = D_3 > 0;$$

$$C_{44}D_3 - C_{14}^2(C_{22}C_{33} - C_{23}^2) > 0 \Rightarrow -\sqrt{\frac{C_{44}D_3}{C_{22}C_{33} - C_{23}^2}} < C_{14} < \sqrt{\frac{C_{44}D_3}{C_{22}C_{33} - C_{23}^2}}.$$

Only the last inequality involves C_{14} ; the others describe the stability of the orthotropic “root medium”.

Stability of orthotropic media 2

The stability of orthotropic media is best described in terms of dimensionless stiffnesses:

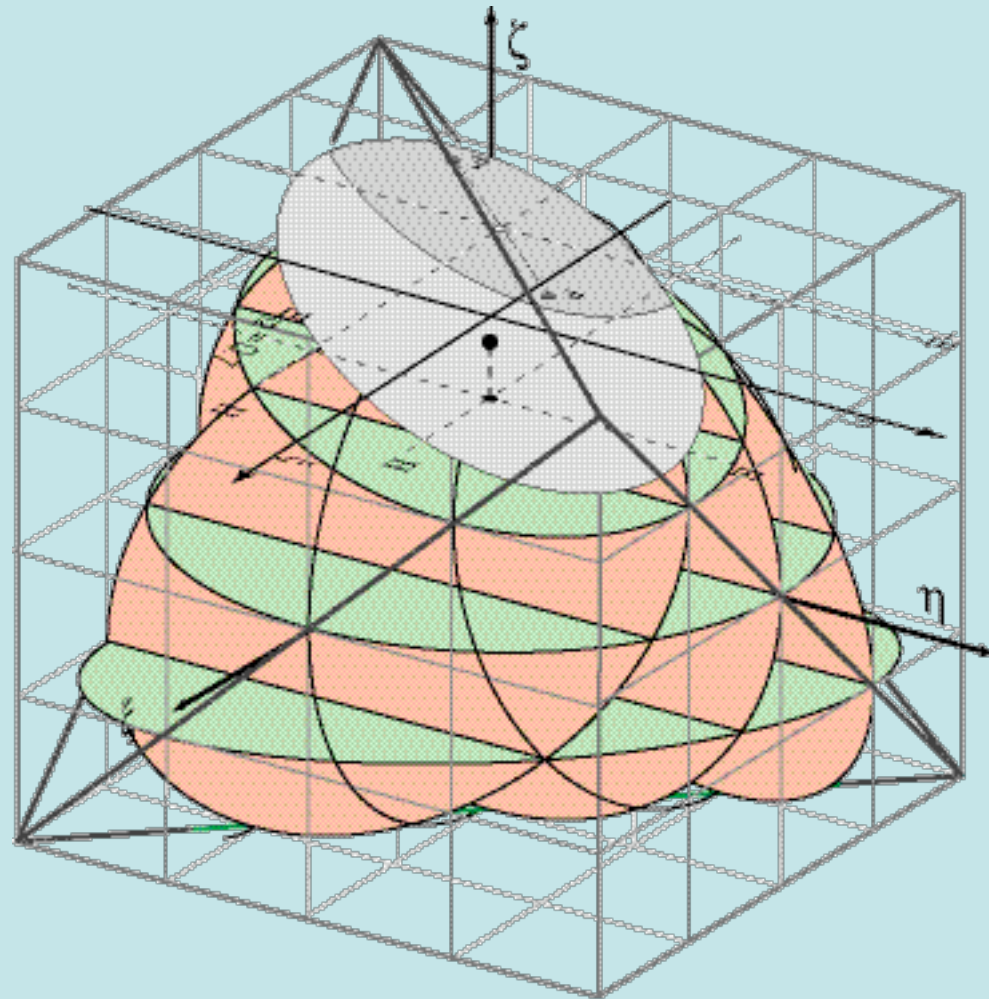
$$\xi^2 = \frac{C_{23}^2}{C_{22}C_{33}}, \quad \eta^2 = \frac{C_{13}^2}{C_{11}C_{33}}, \quad \zeta^2 = \frac{C_{12}^2}{C_{11}C_{22}},$$

The relevant equation of the last slide become

$$\xi^2 < 1, \quad \eta^2 < 1, \quad \zeta^2 < 1, \quad \xi^2 + \eta^2 + \zeta^2 - 2\xi\eta\zeta < 1.$$

The first three inequalities constrain the parameter point $\{\xi, \eta, \zeta\}$ to lie inside the cube with corners $\{\pm 1, \pm 1, \pm 1\}$, and the last equation further constrains it to lie inside the “tetrahedral cushion” inscribed in this cube.

The “tetrahedral cushion”



Constraint volume in parameter space
Planes $\parallel$ to coordinate planes intersect in ellipses

Dimensionless shear stiffnesses

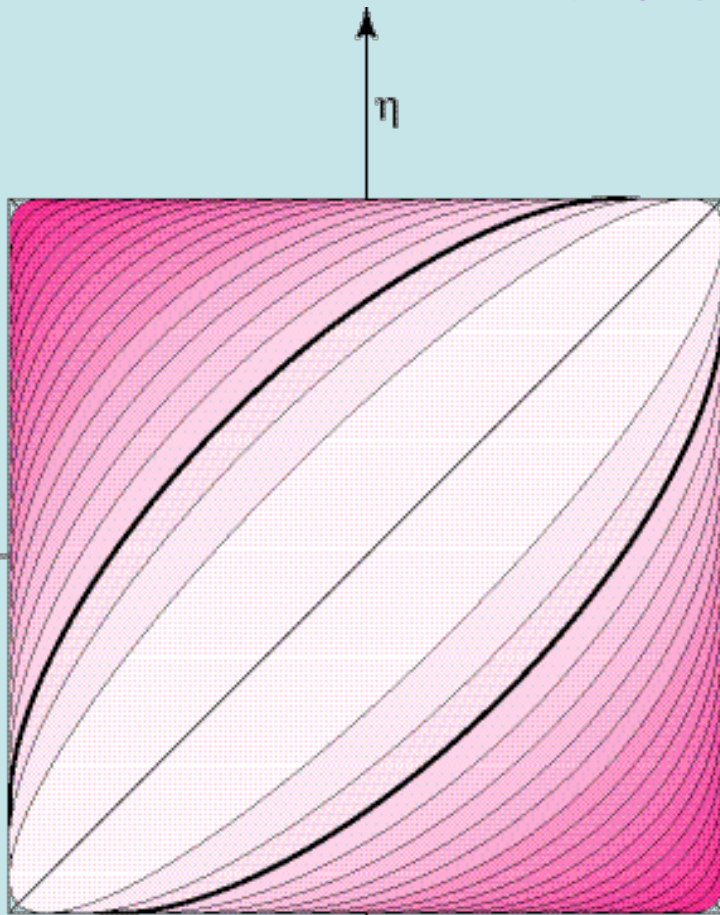
An anomalous companion requires, e.g., $C_{55} + C_{13}^* = -C_{55} + C_{13}$, i.e., $C_{13}^* = -2C_{55} + C_{13}$. To check for the existence of such a solution we normalize the shear stiffnesses in the same way:

$$\Xi^2 = \frac{C_{44}^2}{C_{22}C_{33}}, \quad H^2 = \frac{C_{55}^2}{C_{33}C_{11}}, \quad Z^2 = \frac{C_{66}^2}{C_{11}C_{22}}.$$

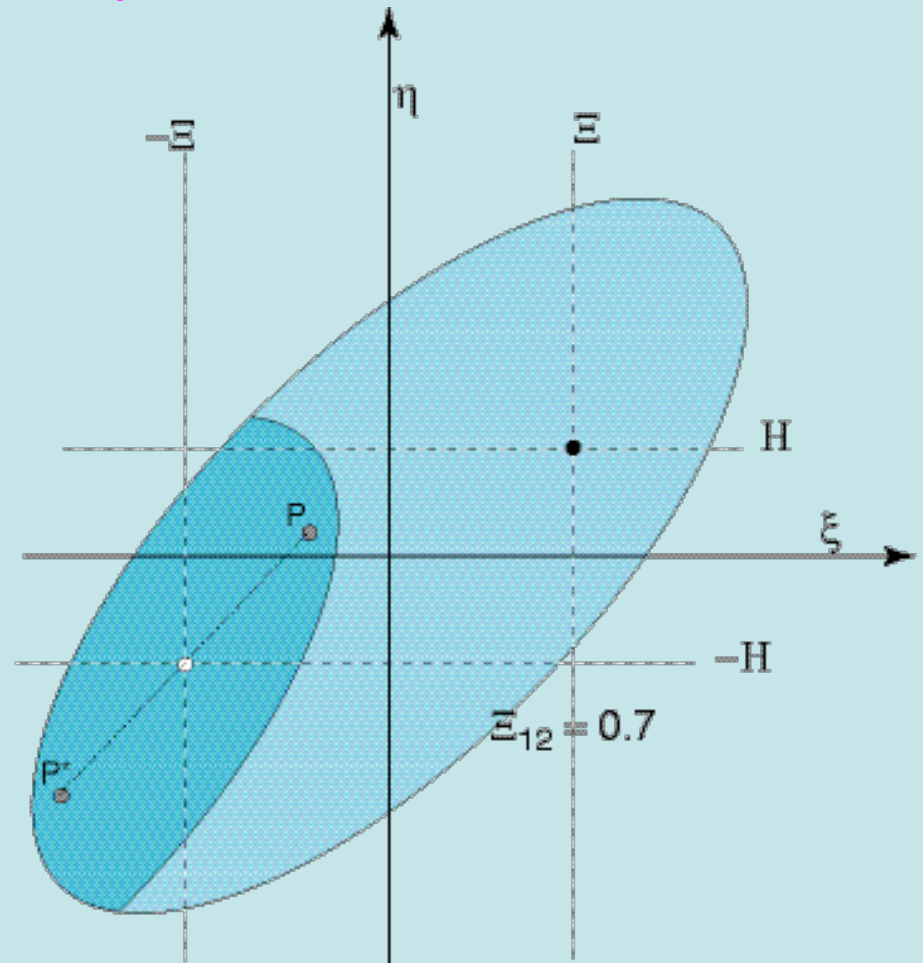
To check for the possibility to have an orthotropic companion with exchange in the 13- and 23-planes, select, e.g., $\xi = C_{12}/\sqrt{(C_{11} C_{22})} = 0.7$. This plane intersects the “cushion” in an ellipse.

Both the point $P = (\xi, \eta, 0.7)$ and the point $P^* = (\xi^*, \eta^*, 0.7)$ must lie in this constraint ellipse, and the point $(-\Xi, -H, 0.7)$ must bisect the line PP^* .

Existence check



“tetrahedral cushion”
select $\zeta = 0.7$



Stability check: both P and P^* lie inside ellipse, (E, H) bisects

Monoclinic extension

Once we are satisfied that a orthotropic root medium with an anomalous companion exists, we can extend the relation to monoclinic symmetry: any C_{16} that satisfies

$$-\sqrt{\frac{C_{66}D_3}{C_{22}C_{33} - C_{23}^2}} < C_{16} < \sqrt{\frac{C_{66}D_3}{C_{22}C_{33} - C_{23}^2}}.$$

will leave the medium stable with either sign.

The Polarization

The components of the polarization vector $\mathbf{p} = p_i$ corresponding to propagation direction $\mathbf{n} = n_i$ and one of the three eigenvalues λ stand in the same ratio as the corresponding cofactors of q_{ji} in the development of $\det(\mathbf{Q}) = \det(\mathbf{\Gamma} - \lambda \mathbf{I})$ for an arbitrary j , i.e.,

$$p_1 : p_2 : p_3 =$$

$$\left((\Gamma_{22} - \lambda)(\Gamma_{33} - \lambda) - \Gamma_{23}^2 \right) : \left(\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda) \right) : \left(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda) \right) \quad \text{for } j = 1,$$

$$\left(\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda) \right) : \left((\Gamma_{11} - \lambda)(\Gamma_{33} - \lambda) - \Gamma_{13}^2 \right) : \left(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda) \right) \quad \text{for } j = 2,$$

$$\left(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda) \right) : \left(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda) \right) : \left((\Gamma_{11} - \lambda)(\Gamma_{22} - \lambda) - \Gamma_{12}^2 \right) \quad \text{for } j = 3.$$

None of these equations is symmetric, thus we should try to symmetrize them.

Symmetric Polarization

It follows from the expressions for $j = 1$ and $j = 3$ that

$$\begin{aligned} & \left(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda) \right) : \left((\Gamma_{11} - \lambda)(\Gamma_{22} - \lambda) - \Gamma_{12}^2 \right) \\ &= \left(\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda) \right) : \left(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda) \right) , \quad \text{or} \\ & \left((\Gamma_{11} - \lambda)(\Gamma_{22} - \lambda) - \Gamma_{12}^2 \right) = \frac{\left(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda) \right) \left(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda) \right)}{\left(\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda) \right)} \end{aligned}$$

After substitution and division of all three terms by

$$\left(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda) \right) \left(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda) \right)$$

one obtains the symmetric expression

$$\begin{aligned} & p_1 : p_2 : p_3 = \\ & \frac{1}{\left(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda) \right)} : \frac{1}{\left(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda) \right)} : \frac{1}{\left(\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda) \right)} \end{aligned}$$

Connection between normal and anomalous polarization

For a pair of companion media with $\Gamma_{12}^* = -\Gamma_{12}$ and $\Gamma_{13}^* = -\Gamma_{13}$ one has

$$p_1 : p_2 : p_3 = \frac{1}{(\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda))} : \frac{1}{(\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda))} : \frac{1}{(\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda))}$$

and

$$p_1^* : p_2^* : p_3^* = \frac{1}{(\Gamma_{12}^*\Gamma_{13}^* - \Gamma_{23}(\Gamma_{11} - \lambda))} : \frac{1}{(\Gamma_{12}^*\Gamma_{23} - \Gamma_{13}^*(\Gamma_{22} - \lambda))} : \frac{1}{(\Gamma_{23}\Gamma_{13}^* - \Gamma_{12}^*(\Gamma_{33} - \lambda))}$$

$$p_1^* : p_2^* : p_3^* = p_1 : -p_2 : -p_3 \quad , \text{ or, since } |\mathbf{p}^*| = |\mathbf{p}| \quad ,$$

$$\mathbf{p}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{p} \quad .$$

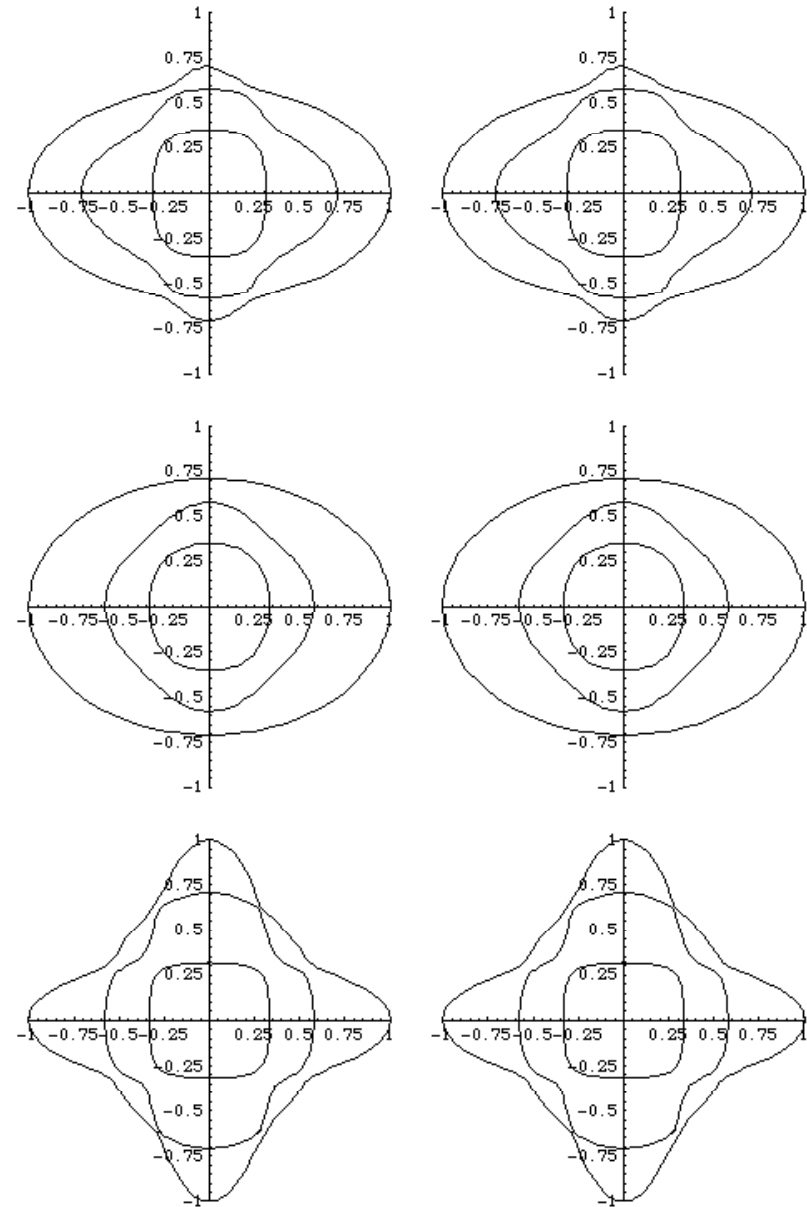
An example

$$\begin{pmatrix} 10 & 2 & 1.5 & 0.8 & 0 & 0 \\ 2 & 9 & 1 & 0 & 0 & 0 \\ 1.5 & 1 & 8 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 10 & -4 & -5.5 & -0.8 & 0 & 0 \\ -4 & 9 & 1 & 0 & 0 & 0 \\ -5.5 & 1 & 8 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Two stiffness matrices with the same slowness surface.

An example 1

The intersection of the slowness surfaces of the two media with the xz , yz , and xy -planes.



An example 2

As in the previous Figure,
but after rotation by $\pi/4$
about the z -axis

