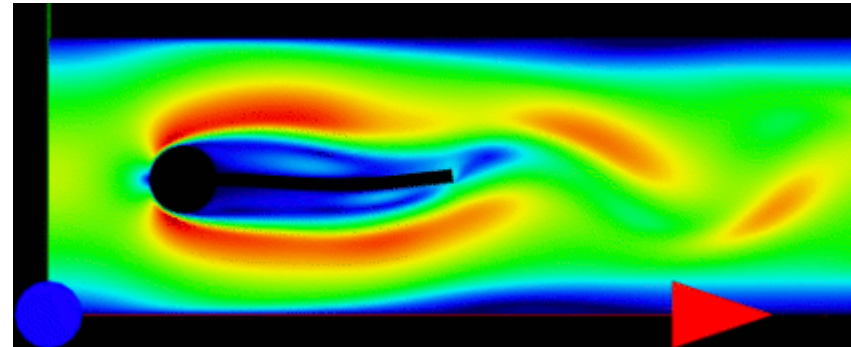


A brief introduction to fluid- structure interactions

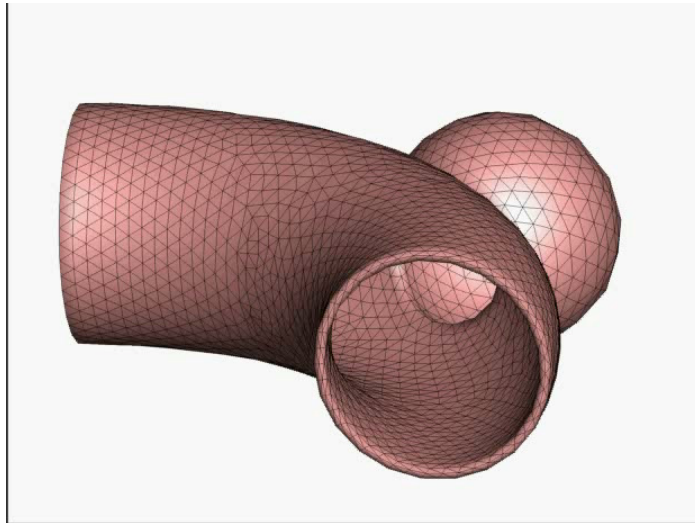
O. Souček

Fluid-structure interactions

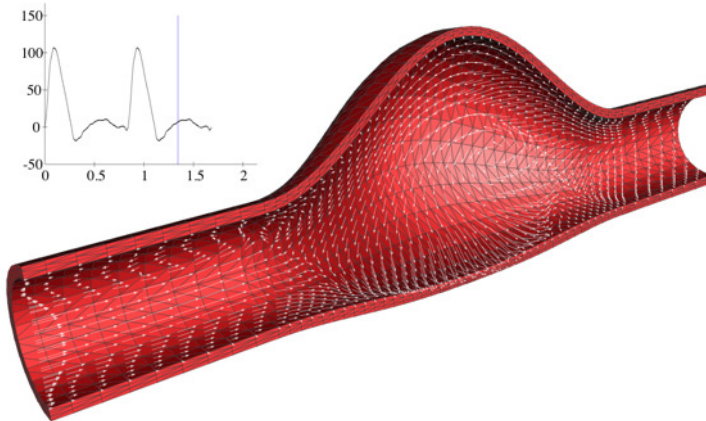
- Important class computational models
- Civil engineering
- Biomechanics
- Industry
- Geophysics
- ...



From <http://www.ihs.uni-stuttgart.de>



M. Madlik



From <https://www.rocq.inria.fr/REO/spip.php?rubrique37>

Fluid-structure interactions

Main technical computational difficulties:

- Typically problems with changing domain
- Interaction – forces exerted by the fluid deform the structure, affecting the motion of the fluid
- Conflict of descriptions fluid – Eulerian vs. structure (solid) – Lagrangian

Two basic computational approaches

Interaction approach

- Fluid and solid parts are solved for iteratively, one by one
- Main advantage – allows the use of traditional solvers for the standard fluid and elasticity problems, the only difficulty is implementation of interaction (convergence)

Fluid-structure interactions

Two basic computational approaches

Interaction approach

- Fluid and solid parts are solved for iteratively, one by one
- Weak coupling (fluid and solid at a time) vs. strong coupling (sub timelevel interactions)
- Main advantage – allows the use of traditional solvers for the standard fluid and elasticity problems, the only difficulty is implementation of interaction (convergence)
- Disadvantage – slow, harder to parallelize (sequential computation), convergence issues

Monolithic approach

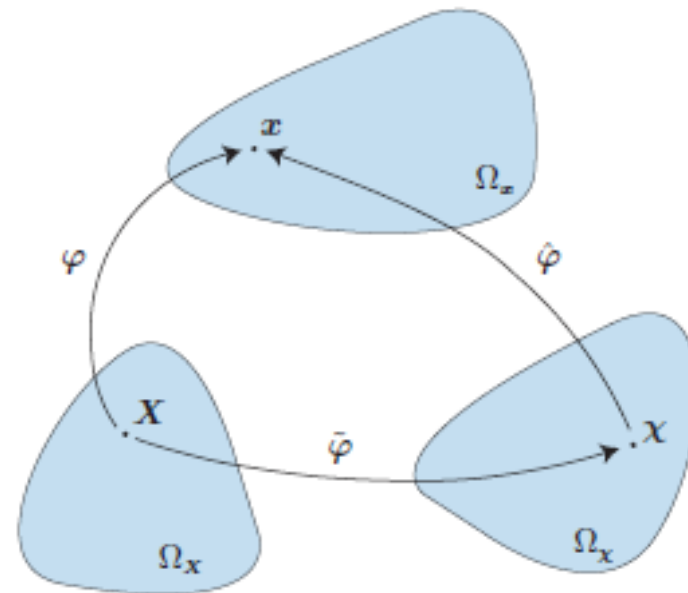
- Both fluid and solid equations are formulated and solved for together
- Advantage – unified approach, parallelizable, allows also for unified space-time discretization
- Disadvantage – typically very large both linear and non-linear problems, hard to treat numerically

Continuum mechanics in moving domains

From: Guglielmo Scovazzi, Thomas J.R. Hughes Lecture notes on Continuum Mechanics on Arbitrary Moving Domains

(http://www.cs.sandia.gov/~gscovaz/publications/CMAMD_SANDarticle.pdf)

- Eulerian (present) configuration
- Lagrangian configuration
- Referential configuration – arbitrarily moving v.r.t. E. and L. conf.



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Lagrangian-to-Eulerian map φ

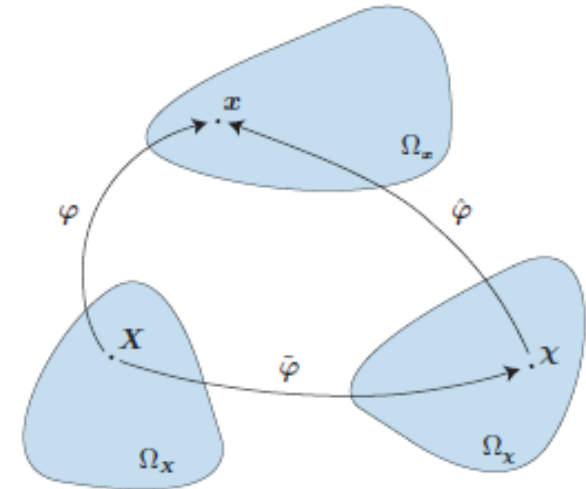
$$\begin{aligned}\varphi(\cdot, t) : \Omega_X &\rightarrow \Omega_x = \varphi(\Omega_X, t), \quad \forall t \geq 0, \\ X &\mapsto x = \varphi(X, t), \quad \forall X \in \Omega_X.\end{aligned}$$

displacement $u = \varphi(X, t) - \varphi(X, 0)$
 $= \varphi(X, t) - X,$

velocity $v = \left. \frac{\partial \varphi}{\partial t} \right|_X = \dot{\varphi}.$

Deformation
gradient,
jacobian

$$\begin{aligned}F &= \nabla_X \varphi = \frac{\partial \varphi}{\partial X} \\ &= \nabla_X x = \frac{\partial x}{\partial X}, \\ J &= \det(F),\end{aligned}$$



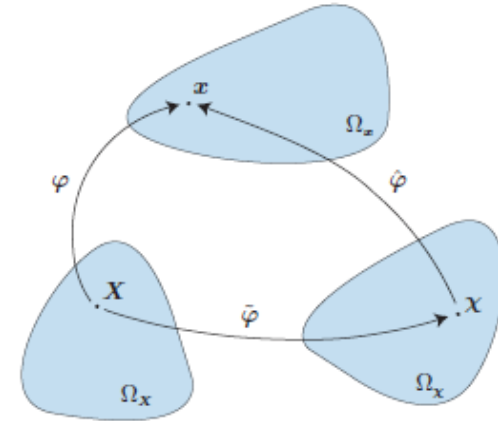
$$\begin{aligned}\dot{\alpha}(x, t) &= \left. \frac{\partial \alpha(x, t)}{\partial t} \right|_X \\ &= \left. \frac{\partial \alpha(\varphi(X, t), t)}{\partial t} \right|_X \\ &= \left. \frac{\partial \alpha}{\partial t} \right|_x + \nabla_x \alpha \cdot \left. \frac{\partial \varphi}{\partial t} \right|_X \\ &= \left. \frac{\partial \alpha}{\partial t} \right|_x + v \cdot \nabla_x \alpha.\end{aligned}$$

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Lagrangian-to-Eulerian map – a generalized space-time perspective

$$\begin{aligned} \varphi: Q_X &\rightarrow Q_x = \varphi(Q_X), \text{ with } Q_X = \Omega_X \times [0, T], \\ (X, \tau) &\mapsto (x, t) = \varphi((X, \tau)), \quad \forall (X, \tau) \in Q_X. \end{aligned}$$



Space-time deformational gradient $\mathbf{F} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{v} & \mathbf{F} \end{bmatrix}.$

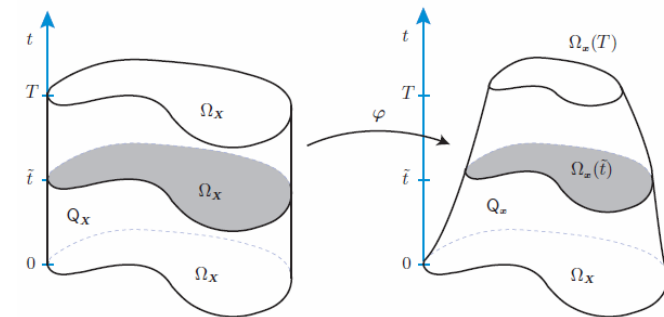
$$\nabla_{[t, X]} = \left\{ \begin{array}{c} \partial_t|_X \\ \nabla_X \end{array} \right\},$$

$$\nabla_{[t, x]} = \left\{ \begin{array}{c} \partial_t|_x \\ \nabla_x \end{array} \right\},$$

$$\nabla_{[t, X]} = \mathbf{F}^T \nabla_{[t, x]}.$$

$$\mathbf{F}^{-1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ -\mathbf{F}^{-1}\mathbf{v} & \mathbf{F}^{-1} \end{bmatrix}.$$

$$J = \det(\mathbf{F}) = J,$$



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Referential-to-Eulerian map $\hat{\varphi}$

$$\hat{\varphi}(\cdot, t) : \Omega_x \rightarrow \Omega_x = \hat{\varphi}(\Omega_x, t), \quad \forall t \geq 0,$$

$$\chi \mapsto x = \hat{\varphi}(\chi, t), \quad \forall \chi \in \Omega_x,$$

Mesh
displacement

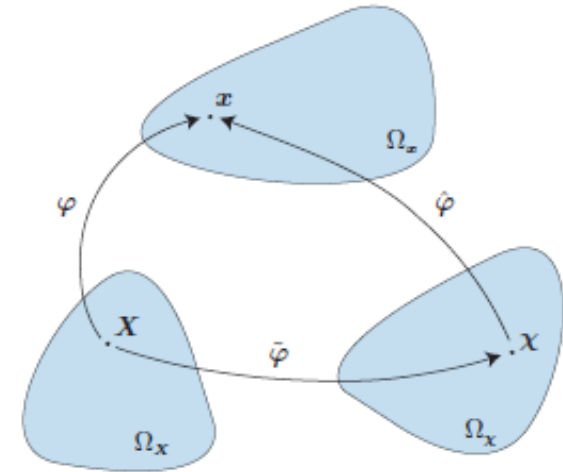
$$\begin{aligned} \hat{u} &= \hat{\varphi}(\chi, t) - \hat{\varphi}(\chi, 0) \\ &= \hat{\varphi}(\chi, t) - \chi, \end{aligned}$$

Mesh velocity

$$\hat{v} = \left. \frac{\partial \hat{\varphi}}{\partial t} \right|_{\chi} = \left. \frac{\partial \hat{u}}{\partial t} \right|_{\chi}.$$

Mesh
deformation
gradient,
Jacobian

$$\begin{aligned} \hat{F} &= \nabla_{\chi} \hat{\varphi} = \frac{\partial \hat{\varphi}}{\partial \chi} \\ &= \nabla_{\chi} x = \frac{\partial x}{\partial \chi}, \\ \hat{J} &= \det(\hat{F}). \end{aligned}$$



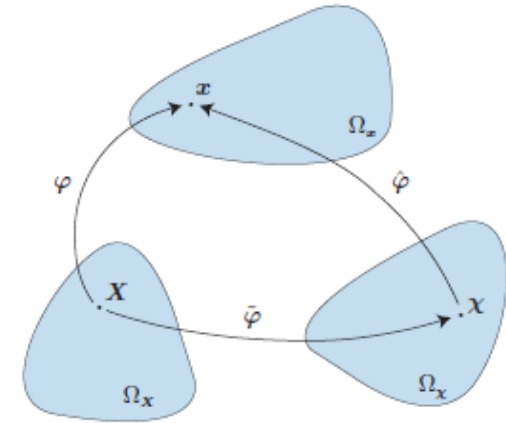
$$\begin{aligned} \left. \frac{\partial \alpha(x, t)}{\partial t} \right|_{\chi} &= \left. \frac{\partial \alpha(\hat{\varphi}(\chi, t), t)}{\partial t} \right|_{\chi} \\ &= \left. \frac{\partial \alpha}{\partial t} \right|_x + \nabla_x \alpha \cdot \left. \frac{\partial \hat{\varphi}}{\partial t} \right|_{\chi} \\ &= \left. \frac{\partial \alpha}{\partial t} \right|_x + \hat{v} \cdot \nabla_x \alpha, \end{aligned}$$

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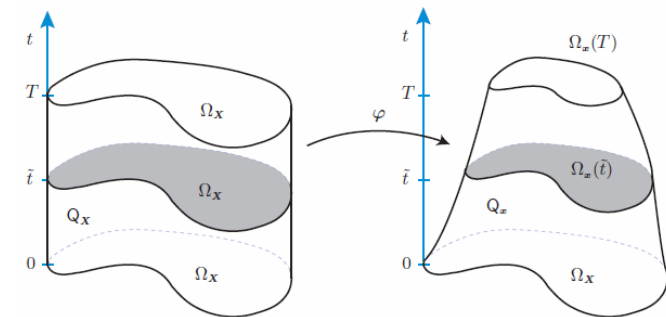
Referential-to-Eulerian map – a generalized space-time perspective

$$\hat{\varphi}: Q_X \rightarrow Q_x = \varphi(Q_X), \quad \text{with } Q_X = \Omega_X \times [0, T], \\ (\chi, t) \mapsto (x, t) = \hat{\varphi}(\chi, t), \quad \forall (\chi, t) \in Q_X.$$



Space-time deformational gradient $\hat{\mathbf{F}} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \hat{\mathbf{v}} & \hat{\mathbf{F}} \end{bmatrix},$

$$\nabla_{[t, \chi]} = \hat{\mathbf{F}}^T \nabla_{[t, x]}, \\ \nabla_{[t, \chi]} = \left\{ \frac{\partial_t|_{\chi}}{\nabla_{\chi}} \right\}. \quad \hat{\mathbf{F}}^{-1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ -\hat{\mathbf{F}}^{-1} \hat{\mathbf{v}} & \hat{\mathbf{F}}^{-1} \end{bmatrix}, \\ \hat{J} = \det(\hat{\mathbf{F}}) = \hat{J}.$$

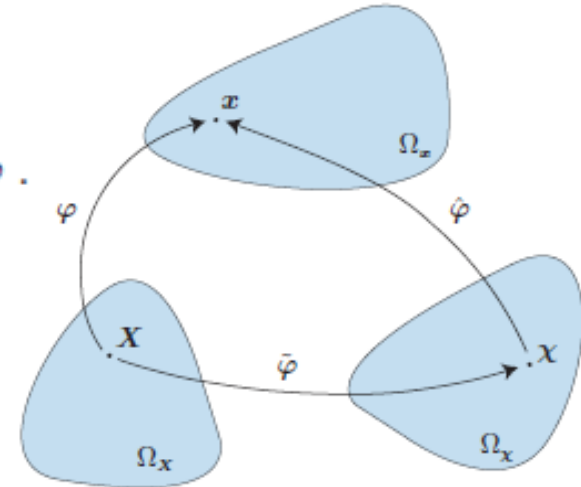


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Lagrangian-to-referential map $\tilde{\varphi} \quad \tilde{\varphi} = \hat{\varphi}^{-1} \circ \varphi .$

$$\begin{aligned} \tilde{\varphi}(\cdot, t) : \Omega_X &\rightarrow \Omega_x = \tilde{\varphi}(\Omega_X, t), \quad \forall t \geq 0, \\ X &\mapsto \chi = \tilde{\varphi}(X, t), \quad \forall X \in \Omega_X . \end{aligned}$$



displacement

$$\begin{aligned} \tilde{u} &= \tilde{\varphi}(X, t) - \tilde{\varphi}(X, 0) \\ &= \tilde{\varphi}(X, t) - X . \end{aligned}$$

velocity

$$\tilde{v} = \left. \frac{\partial \tilde{\varphi}}{\partial t} \right|_X = \left. \frac{\partial \tilde{u}}{\partial t} \right|_X .$$

Deformation
gradient,
jacobian

$$\begin{aligned} \tilde{F} &= \nabla_x \tilde{\varphi} = \frac{\partial \tilde{\varphi}}{\partial X} \\ &= \nabla_x x = \frac{\partial \chi}{\partial X} , \end{aligned}$$

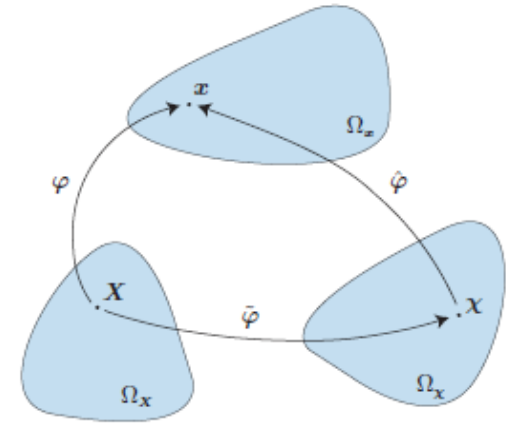
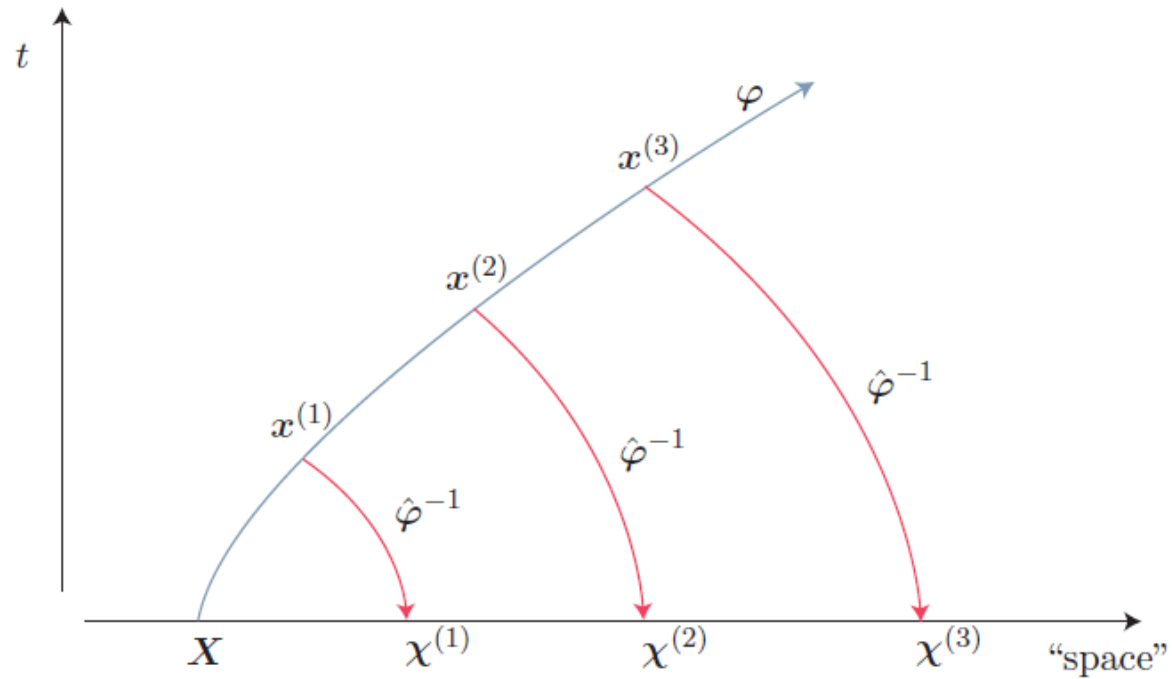
$$\tilde{J} = \det(\tilde{F}) .$$

$$\dot{\alpha}(\chi, t) = \left. \frac{\partial \alpha}{\partial t} \right|_\chi + \tilde{v} \cdot \nabla_x \alpha .$$

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Lagrangian-to-referential map

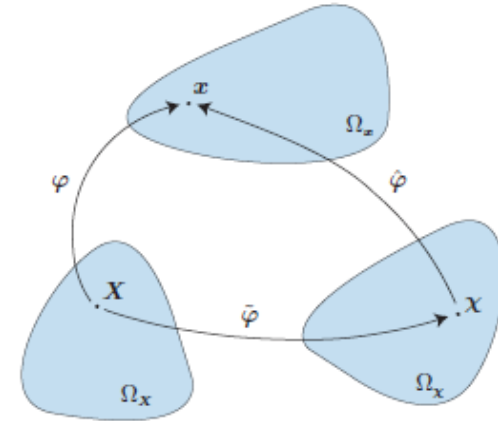


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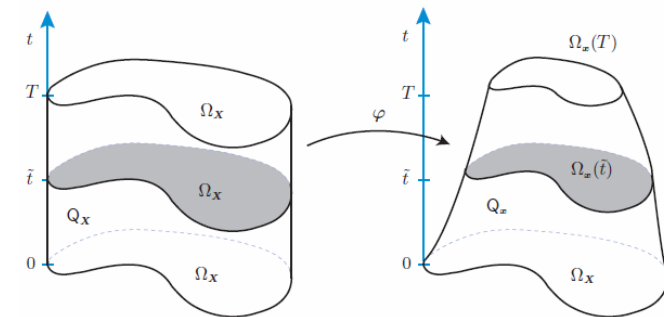
Lagrangian-to-Referential map – a generalized space-time perspective

$$\begin{aligned}\tilde{\varphi}: \quad Q_X &\rightarrow Q_x = \varphi(Q_X), \\ (X, t) &\mapsto (x, t) = \tilde{\varphi}((X, t)), \quad \forall (X, t) \in Q_X.\end{aligned}$$



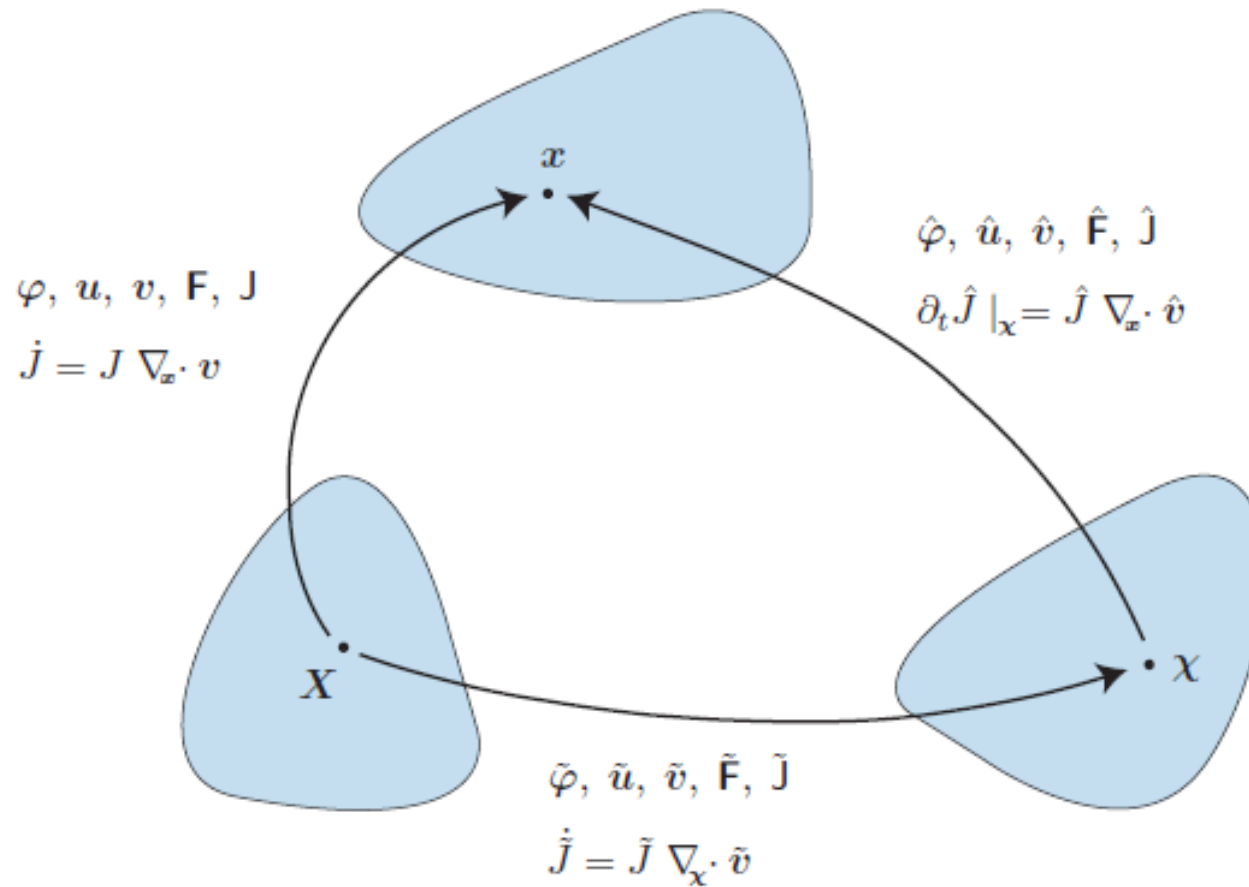
Space-time deformational gradient $\tilde{\mathbf{F}} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \tilde{\mathbf{v}} & \tilde{\mathbf{F}} \end{bmatrix}$

$$\begin{aligned}\tilde{J} &= \det(\tilde{\mathbf{F}}) = \tilde{J}, \\ \tilde{\mathbf{F}}^{-1} &= \begin{bmatrix} 1 & \mathbf{0}^T \\ -\tilde{\mathbf{F}}^{-1}\tilde{\mathbf{v}} & \tilde{\mathbf{F}}^{-1} \end{bmatrix} \\ \nabla_{[t,X]} &= \tilde{\mathbf{F}}^T \nabla_{[t,x]},\end{aligned}$$



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Fundamental kinematic relationships

$$\begin{aligned} \mathbf{v} &= \hat{\mathbf{v}} + \hat{\mathbf{F}} \tilde{\mathbf{v}} , \\ \mathbf{c} &= \mathbf{v} - \hat{\mathbf{v}} = \hat{\mathbf{F}} \tilde{\mathbf{v}} , \end{aligned} \quad \begin{aligned} \dot{\alpha}(\mathbf{x}, t) &= \left. \frac{\partial \alpha}{\partial t} \right|_{\mathbf{x}} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \alpha \\ &= \left. \frac{\partial \alpha}{\partial t} \right|_{\chi} - \hat{\mathbf{v}} \cdot \nabla_{\mathbf{x}} \alpha + \mathbf{v} \cdot \nabla_{\mathbf{x}} \alpha \\ &= \left. \frac{\partial \alpha}{\partial t} \right|_{\chi} + \mathbf{c} \cdot \nabla_{\mathbf{x}} \alpha \end{aligned}$$

$\mathbf{c}$ is the particle velocity relative to the mesh as seen from the Eulerian frame,

$\tilde{\mathbf{v}}$ is the particle velocity seen from the referential frame

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Transport theorems

φ and the classical Reynolds transport theorem

$$\frac{d}{dt} \int_{\Omega_x = \varphi(\Omega_X)} \alpha = \int_{\Omega_x} \frac{\partial \alpha}{\partial t} \Big|_x + \int_{\partial \Omega_x} \alpha \mathbf{v} \cdot \mathbf{n}_x .$$

$\hat{\varphi}$ and the Leibnitz transport theorem

$$\frac{d}{dt} \int_{\Omega_x = \hat{\varphi}(\Omega_X)} \alpha = \int_{\Omega_x} \frac{\partial \alpha}{\partial t} \Big|_x + \int_{\partial \Omega_x} \alpha \hat{\mathbf{v}} \cdot \mathbf{n}_x$$

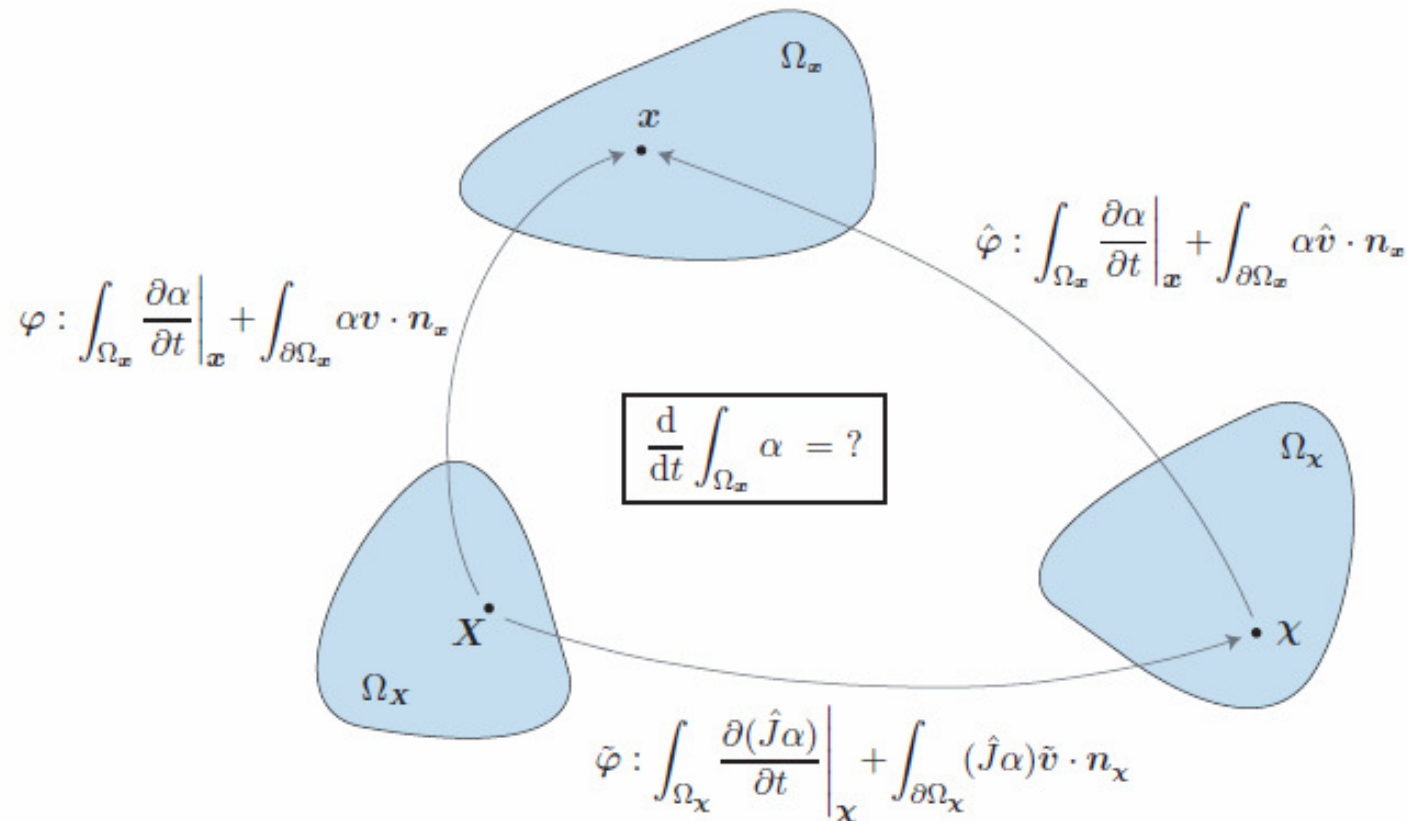
$\tilde{\varphi}$ and the generalized Reynolds transport theorem

$$\frac{d}{dt} \int_{\Omega_x = \tilde{\varphi}(\Omega_X)} \alpha = \int_{\Omega_x} \frac{\partial (\hat{J} \alpha)}{\partial t} \Big|_x + \int_{\partial \Omega_x} (\hat{J} \alpha) \tilde{\mathbf{v}} \cdot \mathbf{n}_x .$$

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Transport theorems



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Transport theorems for surface integrals

$$\frac{d}{dt} \int_{S_x = \varphi(S_X)} \alpha n_x = \int_{S_x} \left(\left. \frac{\partial \alpha}{\partial t} \right|_x + \nabla_x \cdot (\alpha v) \right) n_x - \int_{S_x} \alpha (\nabla_x v)^T n_x$$

$$\frac{d}{dt} \int_{S_x = \hat{\varphi}(S_X)} \alpha n_x = \int_{S_x} \left(\left. \frac{\partial \alpha}{\partial t} \right|_x + \nabla_x \cdot (\alpha \hat{v}) \right) n_x - \int_{S_x} \alpha (\nabla_x \hat{v})^T n_x$$

Transport theorems for line integrals

$$\frac{d}{dt} \oint_{C_x = \varphi(C_X)} \alpha \, dx = \oint_{C_x} \dot{\alpha} \, dx + \oint_{C_x} \alpha (\nabla_x v) \, dx ,$$

$$\frac{d}{dt} \oint_{C_x = \hat{\varphi}(C_X)} \alpha \, dx = \oint_{C_x} \left. \frac{\partial \alpha}{\partial t} \right|_x \, dx + \oint_{C_x} \alpha (\nabla_x \hat{v}) \, dx .$$

Continuum mechanics in moving domains – MASTER BALANCE EQUATIONS

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$$\frac{d}{dt} \int_{\Omega_x = \varphi(\Omega_X)} \alpha = \int_{\partial\Omega_x} \gamma \cdot n_x + \int_{\Omega_x} \beta, \quad \frac{d}{dt} \int_{\Omega_x = \varphi(\Omega_X)} \alpha = \int_{\partial\Omega_x} \Gamma n_x + \int_{\Omega_x} \beta,$$

where the vector γ is the flux of the conserved variable α , and β represents a source/sink term.

Using Reynold's t.t.:

$$\begin{aligned} \int_{\Omega_x = \varphi(\Omega_X)} \left. \frac{\partial \alpha}{\partial t} \right|_x &= \int_{\partial\Omega_x} (\gamma - \alpha v) \cdot n_x + \int_{\Omega_x} \beta, \\ \int_{\Omega_x = \varphi(\Omega_X)} \left. \frac{\partial \alpha}{\partial t} \right|_x &= \int_{\partial\Omega_x} (\Gamma - \alpha \otimes v) n_x + \int_{\Omega_x} \beta, \end{aligned}$$

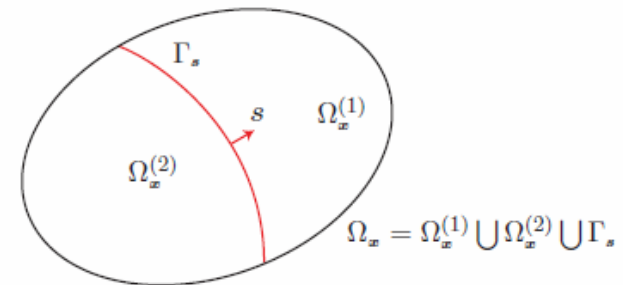
$$\begin{aligned} 0 &= \int_{\Omega_x = \varphi(\Omega_X)} \left. \frac{\partial \alpha}{\partial t} \right|_x + \nabla_x \cdot (\alpha v - \gamma) - \beta, \\ 0 &= \int_{\Omega_x = \varphi(\Omega_X)} \left. \frac{\partial \alpha}{\partial t} \right|_x + \nabla_x \cdot (\alpha \otimes v - \Gamma) - \beta. \end{aligned}$$

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Rankine-Hugoniot conditions

$$[[\alpha]] \cdot s = [[\alpha v - \gamma]] \cdot s, \quad [[\alpha]] s = [[\alpha \otimes v - \Gamma]] \cdot s$$



Master balance law using the Leibnitz transport theorem

$$\int_{\Omega_x = \hat{\varphi}(\Omega_\chi)} \hat{j}^{-1} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_\chi = \int_{\partial\Omega_x} (\gamma - \alpha c) \cdot n_x + \int_{\Omega_x} \beta,$$

$$\int_{\Omega_x = \hat{\varphi}(\Omega_\chi)} \hat{j}^{-1} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_\chi = \int_{\partial\Omega_x} (\Gamma - \alpha \otimes c) \cdot n_x + \int_{\Omega_x} \beta,$$

$$0 = \int_{\Omega_x = \hat{\varphi}(\Omega_\chi)} \hat{j}^{-1} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_\chi + \nabla_x \cdot (\alpha c - \gamma) - \beta,$$

$$0 = \int_{\Omega_x = \hat{\varphi}(\Omega_\chi)} \hat{j}^{-1} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_\chi + \nabla_x \cdot (\alpha \otimes c - \Gamma) - \beta.$$

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Master balance law using the generalized Reynolds transport theorem

$$\int_{\Omega_x} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_x = \int_{\partial\Omega_x} \left(\hat{J}\hat{\mathbf{F}}^{-1}(\gamma - \alpha c) \right) \cdot \mathbf{n}_x + \int_{\Omega_x} \hat{J}\beta ,$$

$$\int_{\Omega_x} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_x = \int_{\partial\Omega_x} (\mathbf{\Gamma} - \alpha \otimes c) \hat{J}\hat{\mathbf{F}}^{-T} \mathbf{n}_x + \int_{\Omega_x} \hat{J}\beta ,$$

$$0 = \int_{\Omega_x} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_x + \nabla_x \cdot \left(\hat{J}\hat{\mathbf{F}}^{-1}(\alpha c - \gamma) \right) - \hat{J}\beta ,$$

$$0 = \int_{\Omega_x} \frac{\partial(\hat{J}\alpha)}{\partial t} \bigg|_x + \nabla_x \cdot \left((\alpha \otimes c - \mathbf{\Gamma}) \hat{J}\hat{\mathbf{F}}^{-T} \right) - \hat{J}\beta .$$

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A generalized space-time approach to the master balance laws

$$0 = \int_{\Omega_x = \varphi(\Omega_X)} \frac{\partial \alpha}{\partial t} \Big|_x + \nabla_x \cdot (\alpha v - \gamma) - \beta ,$$

may be written as

$$0 = \int_0^t \int_{\Omega_x = \varphi(\Omega_X)} \nabla_{[t,x]} \cdot \gamma_{[t,x]} - \beta , \quad \text{with} \quad \gamma_{[t,x]} = \left\{ \begin{array}{c} \alpha \\ \alpha v - \gamma \end{array} \right\}$$

$$\begin{aligned} 0 &= \int_{Q_x = \varphi(Q_X)} \nabla_{[t,x]} \cdot \gamma_{[t,x]} - \beta \\ &= \int_{Q_X} (J F^{-T} \nabla_{[t,X]}) \cdot \gamma_{[t,x]} - J \beta \\ &= \int_{Q_X} \nabla_{[t,X]} \cdot (J F^{-1} \gamma_{[t,x]}) - J \beta \\ &= \int_{Q_X} \nabla_{[t,X]} \cdot \gamma_{[t,X]} - J \beta , \end{aligned}$$

where

$$\begin{aligned} \gamma_{[t,X]} &= J F^{-1} \gamma_{[t,x]} \\ &= J \begin{bmatrix} 1 & \mathbf{0}^T \\ -F^{-1}v & F^{-1} \end{bmatrix} \left\{ \begin{array}{c} \alpha \\ \alpha v - \gamma \end{array} \right\} \\ &= \left\{ \begin{array}{c} J\alpha \\ -JF^{-1}\gamma \end{array} \right\} . \end{aligned}$$

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A generalized space-time approach to the master balance laws

Space-time master balance law using the $\hat{\phi}$ map

$$0 = \int_{\Omega_x} \frac{\partial(\hat{J}\alpha)}{\partial t} \Big|_x + \nabla_x \cdot \left(\hat{J}\hat{\mathbf{F}}^{-1}(\alpha c - \gamma) \right) - \hat{J}\beta, \quad \text{may be written as}$$

$$0 = \int_{Q_x} \nabla_{[t,x]} \cdot \gamma_{[t,x]} - \hat{J}\beta, \quad \text{with} \quad \gamma_{[t,x]} = \left\{ \begin{array}{l} \hat{J}\alpha \\ \hat{J}\hat{\mathbf{F}}^{-1}(\alpha c - \gamma) \end{array} \right\}.$$

$$\begin{aligned} 0 &= \int_{Q_x} \nabla_{[t,x]} \cdot \gamma_{[t,x]} - \hat{J}\beta \\ &= \int_{Q_x} \nabla_{[t,x]} \cdot (\hat{J}\hat{\mathbf{F}}^{-1} \gamma_{[t,x]}) - \hat{J}\beta \\ &= \int_{Q_x} \left(\hat{J}\hat{\mathbf{F}}^{-T} \nabla_{[t,x]} \right) \cdot \gamma_{[t,x]} - \hat{J}\beta \\ &= \int_{Q_x = \hat{\phi}(Q_x)} \nabla_{[t,x]} \cdot \gamma_{[t,x]} - \beta, \end{aligned}$$

Continuum mechanics in moving domains – MASTER BALANCE EQUATIONS

From: Guglielmo Scovazzi, Thomas J.R. Hughes Lecture notes on
Continuum Mechanics on Arbitrary Moving Domains
(http://www.cs.sandia.gov/~gscovaz/publications/CMAMD_SANDarticle.pdf)

A generalized space-time approach to the master balance laws

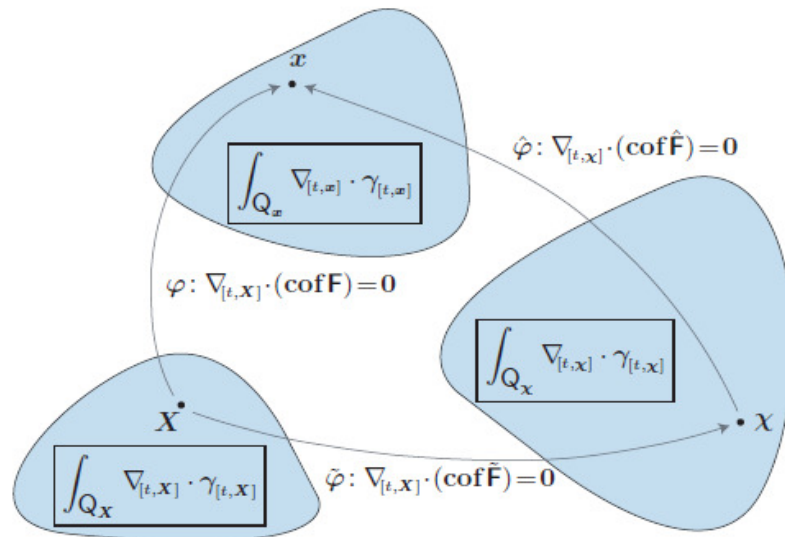
Space-time master balance law using the $\tilde{\varphi}$ map

$$\begin{aligned} 0 &= \int_{Q_x} \nabla_{[t,x]} \cdot \gamma_{[t,x]} - \hat{J}\beta \\ &= \int_{Q_X} \nabla_{[t,X]} \cdot \gamma_{[t,X]} - J\beta . \end{aligned}$$

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[e.pdf](#))

Conservation equation	α / α	β / β	γ / Σ
Mass	ρ	0	0
Momentum	ρv	ρb	σ
Angular momentum	$x \times \rho v$	$x \times \rho b$	$x \times \sigma = \varepsilon_{ikl} x_k \sigma_{lj}$
Total energy	$\rho e + \rho v \cdot v / 2$	$\rho r + \rho v \cdot b$	$\sigma^T v - q$
Entropy	$\rho \eta$	$\rho r / \theta$	$-q / \theta$



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Conservation of mass

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial(\hat{J}\rho)}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot (\hat{J}\hat{\mathbf{F}}^{-1}\rho\mathbf{c})$$

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial(\hat{J}\rho)}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot (\hat{J}\rho\tilde{\mathbf{v}}) .$$

$$0 = \int_{\Omega_{\mathbf{x}}=\hat{\varphi}(\Omega_{\mathbf{X}})} \hat{J}^{-1} \frac{\partial(\hat{J}\rho)}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot (\rho\mathbf{c}) .$$

Conservation of momentum

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial(\hat{J}\rho\mathbf{v})}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot ((\rho\mathbf{v} \otimes \mathbf{c} - \boldsymbol{\sigma})\hat{J}\hat{\mathbf{F}}^{-T}) + \hat{J}\rho\mathbf{b}$$

$\hat{\mathbf{P}} = \hat{J}\boldsymbol{\sigma}\hat{\mathbf{F}}^{-T}$ The Piola-Kirchhoff stress tensor

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial(\hat{J}\rho\mathbf{v})}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot ((\hat{J}\rho\mathbf{v}) \otimes \tilde{\mathbf{v}} - \hat{\mathbf{P}}) + \hat{J}\rho\mathbf{b}$$

$$0 = \int_{\Omega_{\mathbf{x}}=\hat{\varphi}(\Omega_{\mathbf{X}})} \hat{J}^{-1} \frac{\partial(\hat{J}\rho\mathbf{v})}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot (\rho\mathbf{v} \otimes \mathbf{c} - \boldsymbol{\sigma}) + \rho\mathbf{b}$$

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Conservation of angular momentum

$$0 = \int_{\Omega_{\chi}} \frac{\partial(\hat{J} \mathbf{x} \times \rho \mathbf{v})}{\partial t} \Big|_{\chi} + \nabla_{\mathbf{x}} \cdot \left((\mathbf{x} \times \rho \mathbf{v}) \otimes \mathbf{c} - \mathbf{x} \times \boldsymbol{\sigma} \right) \hat{J} \hat{\mathbf{F}}^{-T} + \hat{J} \mathbf{x} \times \rho \mathbf{b}$$

Conservation of total energy

$$0 = \int_{\Omega_{\chi}} \frac{\partial(\hat{J} \rho E)}{\partial t} \Big|_{\chi} + \nabla_{\mathbf{x}} \cdot \left(\hat{J} \hat{\mathbf{F}}^{-1} (\rho E \mathbf{c} - \boldsymbol{\sigma}^T \mathbf{v} + \mathbf{q}) \right) + \hat{J} \rho (\mathbf{b} \cdot \mathbf{v} + r)$$

$$0 = \int_{\Omega_{\chi}} \frac{\partial(\hat{J} \rho E)}{\partial t} \Big|_{\chi} + \nabla_{\mathbf{x}} \cdot \left((\hat{J} \rho E) \tilde{\mathbf{v}} - \hat{\mathbf{P}}^T \mathbf{v} + \hat{\mathbf{Q}} \right) + \hat{J} \rho (\mathbf{b} \cdot \mathbf{v} + r)$$

$$0 = \int_{\Omega_{\mathbf{x}} = \hat{\varphi}(\Omega_{\chi})} \hat{J}^{-1} \frac{\partial(\hat{J} \rho E)}{\partial t} \Big|_{\chi} + \nabla_{\mathbf{x}} \cdot (\rho E \mathbf{c} - \boldsymbol{\sigma}^T \mathbf{v} + \mathbf{q}) + \rho (\mathbf{b} \cdot \mathbf{v} + r)$$

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Entropy balance

$$\begin{aligned}
 0 &\leq \int_{\Omega_{\mathbf{x}}} \frac{\partial(\hat{J}\rho\eta)}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot \left(\hat{J}\hat{\mathbf{F}}^{-1} \left(\rho\eta\mathbf{c} + \frac{\mathbf{q}}{\theta} \right) \right) + \hat{J}\rho\frac{r}{\theta} \\
 &= \int_{\Omega_{\mathbf{x}}} \frac{\partial(\hat{J}\rho\eta)}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot \left(\left(\hat{J}\rho\eta \right) \tilde{\mathbf{v}} + \frac{\hat{\mathbf{Q}}}{\theta} \right) + \hat{J}\rho\frac{r}{\theta} , \\
 0 &\leq \int_{\Omega_{\mathbf{x}}=\hat{\varphi}(\Omega_{\hat{\mathbf{x}}})} \hat{J}^{-1} \frac{\partial(\hat{J}\rho\eta)}{\partial t} \Big|_{\mathbf{x}} + \nabla_{\mathbf{x}} \cdot \left(\rho\eta\mathbf{c} + \frac{\mathbf{q}}{\theta} \right) + \rho\frac{r}{\theta} .
 \end{aligned}$$

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Equations in conservative vector form

$$\mathbf{U} = \begin{Bmatrix} \rho \\ \rho \mathbf{v} \\ \rho E \end{Bmatrix}, \quad \mathbf{G} = \mathbf{U} \otimes \mathbf{c} - \begin{bmatrix} \mathbf{0}^T \\ \boldsymbol{\sigma} \\ \mathbf{v}^T \boldsymbol{\sigma} - q^T \end{bmatrix}, \quad \mathbf{Z} = \begin{Bmatrix} \mathbf{0}^T \\ \rho \mathbf{b} \\ \rho(\mathbf{b} \cdot \mathbf{v} + r) \end{Bmatrix}$$

$$\mathbf{0} = \int_{\Omega_{\chi}} \partial_t|_{\chi}(\hat{J}\mathbf{U}) + \nabla_{\chi} \cdot (\hat{J}\mathbf{G}\hat{\mathbf{F}}^{-T}) + \hat{J}\mathbf{Z}$$

$$\mathbf{0} = \int_{\Omega_{\mathbf{x}} = \hat{\varphi}(\Omega_{\chi})} \hat{J}^{-1} \partial_t|_{\chi}(\hat{J}\mathbf{U}) + \nabla_{\mathbf{x}} \cdot \mathbf{G} + \mathbf{Z}$$

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Space-time conservative vector form

$$0 = \int_{Q_\chi} \nabla_{[t,\chi]} \cdot \mathbf{G}_{[t,\chi]} + \hat{\mathbf{J}} \mathbf{Z} ,$$

$$\mathbf{G}_{[t,\chi]} = \left[\hat{\mathbf{J}} \mathbf{U} \mid \hat{\mathbf{J}} \mathbf{G} \hat{\mathbf{F}}^{-T} \right] \quad \mathbf{G}_{[t,\chi]} = \hat{\mathbf{J}} \mathbf{G}_{[t,\mathbf{x}]} \hat{\mathbf{F}}^{-T}$$

$$\mathbf{G}_{[t,\mathbf{x}]} = \left[\mathbf{U} \mid \mathbf{U} \mathbf{v}^T - \begin{bmatrix} \mathbf{0}^T \\ \boldsymbol{\sigma} \\ \mathbf{v}^T \boldsymbol{\sigma} - q \end{bmatrix} \right]$$

$$0 = \int_{Q_{\mathbf{x}} = \hat{\varphi}(Q_\chi)} \nabla_{[t,\mathbf{x}]} \cdot \mathbf{G}_{[t,\mathbf{x}]} + \mathbf{Z} .$$

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Advective form of the mass conservation equation

$$\int_{\Omega_{\chi}} \hat{J} \left. \frac{\partial \rho}{\partial t} \right|_{\chi} + \hat{J} \tilde{v} \cdot \nabla_{\chi} \rho + \hat{J} \rho \hat{F}^{-T} : \nabla_{\chi} v$$

$$0 = \int_{\Omega_{\mathbf{x}} = \hat{\varphi}(\Omega_{\chi})} \left. \frac{\partial \rho}{\partial t} \right|_{\chi} + c \cdot \nabla_{\mathbf{x}} \rho + \rho \nabla_{\mathbf{x}} \cdot v \quad \dot{\rho} = \left. \frac{\partial \rho}{\partial t} \right|_{\chi} + c \cdot \nabla_{\mathbf{x}} \rho$$

Advective form of the momentum equation

$$0 = \int_{\Omega_{\chi}} \hat{J} \rho \left. \frac{\partial v}{\partial t} \right|_{\chi} + \hat{J} \rho (\nabla_{\chi} v) \tilde{v} - \nabla_{\chi} \cdot \hat{P} + \hat{J} \rho b$$

$$0 = \int_{\Omega_{\mathbf{x}} = \hat{\varphi}(\Omega_{\chi})} \rho \left. \frac{\partial v}{\partial t} \right|_{\chi} + \rho (\nabla_{\mathbf{x}} v) c - \nabla_{\mathbf{x}} \cdot \sigma + \rho b$$

Continuum mechanics in moving domains – CONSERVATION LAWS

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Advective form of the total energy equation

$$\begin{aligned} 0 &= \int_{\Omega_x} \frac{\partial(\hat{J}\rho E)}{\partial t} \bigg|_x + \nabla_x \cdot (\hat{J}\hat{F}^{-1}(\rho E c - \sigma^T v + q)) + \hat{J}\rho(b \cdot v + r) \\ &= \int_{\Omega_x} \hat{J}\rho \frac{\partial e}{\partial t} \bigg|_x + \hat{J}\rho \tilde{v} \nabla_x e - (\hat{J}\sigma F^{-T}) : \nabla_x v + \nabla_x \cdot (\hat{J}F^{-1}q) + \hat{J}\rho r \end{aligned}$$

$$0 = \int_{\Omega_x} \hat{J}\rho \frac{\partial e}{\partial t} \bigg|_x + \hat{J}\rho \tilde{v} \nabla_x e - \hat{P} : \nabla_x v + \nabla_x \cdot \hat{Q} + \hat{J}\rho r$$

$$0 = \int_{\Omega_x = \hat{\varphi}(\Omega_\chi)} \rho \frac{\partial e}{\partial t} \bigg|_x + c \cdot \nabla_x e - \sigma : \nabla_x v + \nabla_x \cdot q + \rho r$$

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Advective form of the entropy equation

$$\begin{aligned}
 0 &\leq \int_{\Omega_x} \frac{\partial(\hat{J}\rho\eta)}{\partial t} \Big|_x + \nabla_x \cdot \left(\hat{J}\hat{\mathbf{F}}^{-1} \left(\rho\eta\mathbf{c} + \frac{\mathbf{q}}{\theta} \right) \right) + \hat{J}\rho\frac{r}{\theta} \\
 &= \int_{\Omega_x} \hat{J}\rho \frac{\partial\eta}{\partial t} \Big|_x + \hat{J}\rho\tilde{\mathbf{v}}\nabla_x\eta + \nabla_x \cdot \left(\frac{\hat{\mathbf{Q}}}{\theta} \right) + \hat{J}\rho\frac{r}{\theta} \\
 0 &\leq \int_{\Omega_x} \rho \frac{\partial\eta}{\partial t} \Big|_x + \rho\mathbf{c} \cdot \nabla_x\eta + \nabla_x \cdot \left(\frac{\mathbf{q}}{\theta} \right) + \rho\frac{r}{\theta}
 \end{aligned}$$

Continuum mechanics in moving domains – CONSERVATION LAWS

From: M. Mádlík 2010, Interaction of a Fluid Flow with an Elastic Body,
Ph.D. Thesis

Lagrangian Forms $\mathbf{v} = \hat{\mathbf{v}}, \quad \tilde{\mathbf{v}} = 0, c = 0, \hat{J} = J, \hat{\mathbf{F}} = \mathbf{F}$

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial J \rho}{\partial t},$$

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial (J \rho \mathbf{v})}{\partial t} + \nabla_{\mathbf{x}} \cdot (-\mathbf{P}) + J \rho \mathbf{b}$$

$$0 = \int_{\Omega_{\mathbf{x}}} J \rho \frac{\partial \mathbf{v}}{\partial t} + J \rho (\nabla_{\mathbf{x}} \mathbf{v}) \mathbf{v} - \nabla_{\mathbf{x}} \mathbf{P} + J \rho \mathbf{b},$$

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial (J \rho E)}{\partial t} + \nabla_{\mathbf{x}} \cdot (-\mathbf{P}^T \mathbf{v} + \mathbf{Q}) + J \rho (\mathbf{b} \cdot \mathbf{v} + r).$$

Continuum mechanics in moving domains – CONSERVATION LAWS

From: M. Mádlik 2010, Interaction of a Fluid Flow with an Elastic Body,
Ph.D. Thesis

Eulerian Forms $\hat{v} = 0, \quad c = v, \hat{F} = id, \hat{j} = 1.$

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}),$$

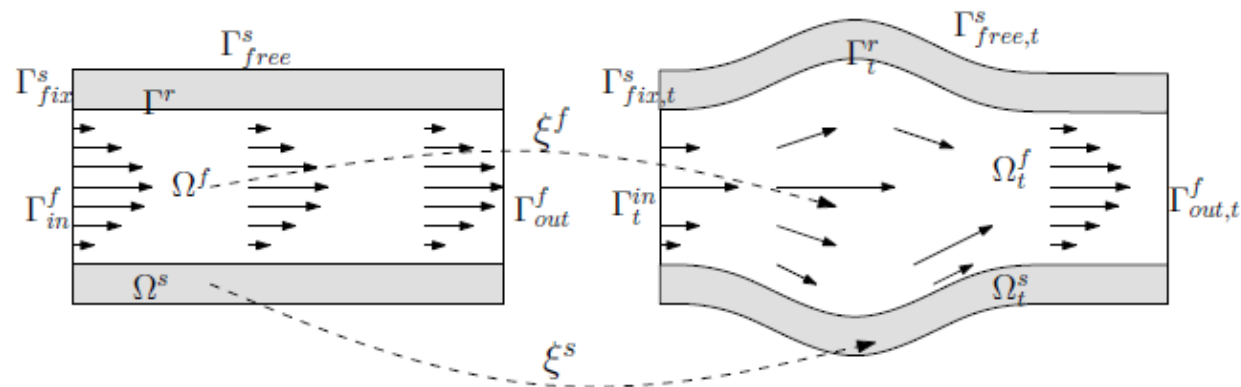
$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla_{\mathbf{x}} \cdot ((\rho \mathbf{v} \otimes \mathbf{v}) - \boldsymbol{\sigma}) + \rho \mathbf{b}$$

$$0 = \int_{\Omega_{\mathbf{x}}} \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\nabla_{\mathbf{x}} \mathbf{v}) \mathbf{v} - \nabla_{\mathbf{x}} \cdot \boldsymbol{\sigma} + \rho \mathbf{b},$$

$$0 = \int_{\Omega_{\mathbf{x}}} \frac{\partial(\rho E)}{\partial t} + \nabla_{\mathbf{x}} \cdot ((\rho E \mathbf{v} - \boldsymbol{\sigma}^T \mathbf{v} + \mathbf{q}) + \rho(\mathbf{b} \cdot \mathbf{v} + r)).$$

Continuum mechanics in moving domains – Single continuum model

From: M. Mádlik 2010, Interaction of a Fluid Flow with an Elastic Body,
Ph.D. Thesis



Solid Part.

$$\xi^s : \Omega^s \times [0, T] \rightarrow \Omega_t^s.$$

$$u^s(X, t) = \xi^s(X, t) - X, \quad v^s(X, t) = \frac{\partial \xi^s}{\partial t}(X, t).$$

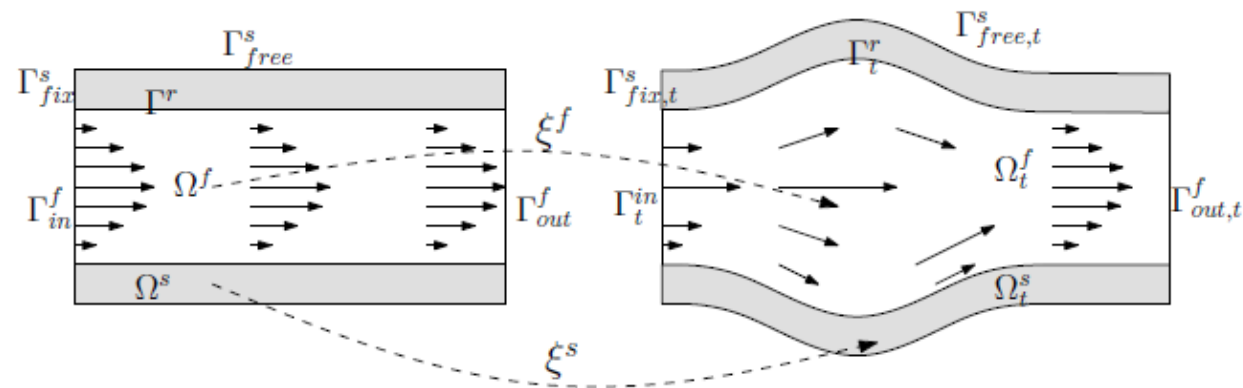
$$0 = \nabla_{\mathbf{x}} \cdot (\hat{J} \mathbf{v}^s \hat{\mathbf{F}}^{-\top}),$$

$$0 = \hat{J} \rho \frac{\partial \mathbf{v}^s}{\partial t} \Big|_{\mathbf{x}} + \hat{J} \rho \nabla_{\mathbf{x}} \mathbf{v}^s (\hat{\mathbf{F}}^{-1} \mathbf{c}) - \nabla_{\mathbf{x}} \cdot (\hat{J} \boldsymbol{\sigma} \hat{\mathbf{F}}^{-\top}).$$

$$\mathbf{v}^s = 0 \quad \text{on } \Gamma_{fix,t}^s \quad \boldsymbol{\sigma}^s \mathbf{n} = 0 \quad \text{on } \Gamma_{free,t}^s.$$

Continuum mechanics in moving domains – Single continuum model

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Fluid Part

$$\xi^f : \Omega^f \times [0, T] \rightarrow \Omega_t^f.$$

$$u^f(X, t) = \xi^f(X, t) - X.$$

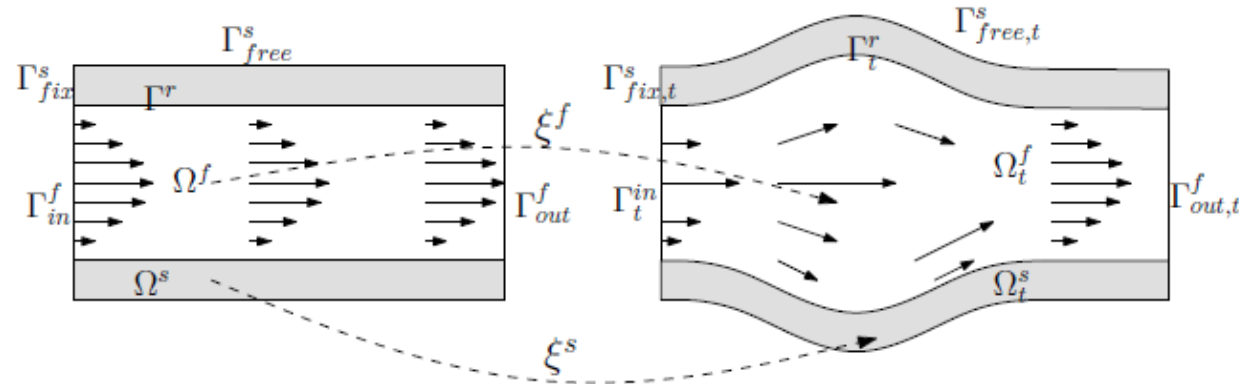
$$0 = \nabla_{\mathbf{x}} \cdot (\hat{J} \mathbf{v}^f \hat{\mathbf{F}}^{-\top}),$$

$$0 = \hat{J} \rho \frac{\partial \mathbf{v}^f}{\partial t} \Big|_{\mathbf{x}} + \hat{J} \rho \nabla_{\mathbf{x}} v^f (\hat{\mathbf{F}}^{-1} \mathbf{c}) - \nabla_{\mathbf{x}} \cdot (\hat{J} \boldsymbol{\sigma} \hat{\mathbf{F}}^{-\top}).$$

$$\boldsymbol{\sigma}^f \mathbf{n} = p_{in} \mathbf{n} \quad \text{on } \Gamma_{in,t}^f, \quad \boldsymbol{\sigma}^f \mathbf{n} = p_{out} \mathbf{n} \quad \text{on } \Gamma_{out,t}^f$$

Continuum mechanics in moving domains – Single continuum model

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Interacting Boundary.

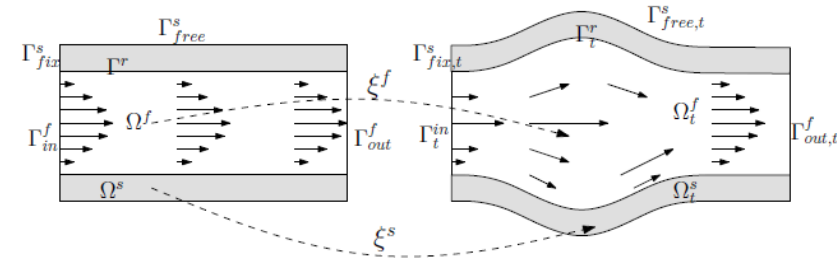
$$\xi^f(X, t) = \xi^s(X, t) \quad \forall (X, t) \in \Gamma^r \times [0, T].$$

$$\sigma^f n_\chi = -\sigma^s n_\chi \quad \text{on } \Gamma_t^r$$

$$v^f = v^s \quad \text{on } \Gamma_t^r.$$

Continuum mechanics in moving domains – Single continuum model

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Ph.D. Thesis



Monolithic Reformulation.

$$\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^3,$$

$$\mathbf{v} = \begin{cases} \mathbf{v}^s & \text{in } \Omega^s \\ \mathbf{v}^f & \text{in } \Omega^f \end{cases}$$

$$\mathbf{v} : \Omega \times [0, T] \rightarrow \mathbb{R}^3,$$

$$\mathbf{u} = \begin{cases} \mathbf{u}^s & \text{in } \Omega^s \\ \mathbf{u}^f & \text{in } \Omega^f. \end{cases}$$

$$\mathbf{F} = \mathbf{I} + \nabla_{\mathbf{x}} \mathbf{u}, \quad J = \det \mathbf{F}.$$

Lagrangian descr. In the solid part

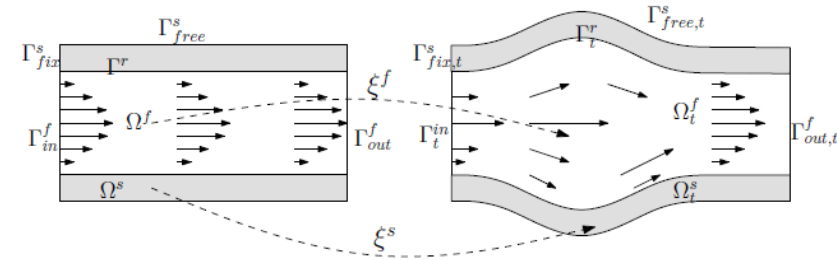
$$\begin{aligned} J &= 1 && \text{in } \Omega^s, \\ J \rho^s \frac{\partial \mathbf{v}}{\partial t} &= \nabla_{\mathbf{x}} \cdot \mathbf{P}^s && \text{in } \Omega^s. \end{aligned}$$

ALE in the fluid part

$$\begin{aligned} 0 &= \nabla_{\mathbf{x}} \cdot (J \mathbf{v} \mathbf{F}^{-\top}), \\ 0 &= J \rho^f \frac{\partial \mathbf{v}}{\partial t} \Big|_{\mathbf{x}} + J \rho^f \nabla_{\mathbf{x}} \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}^f}{\partial t} \right) \right) - \nabla_{\mathbf{x}} \cdot (J \boldsymbol{\sigma} \mathbf{F}^{-\top}). \end{aligned}$$

Continuum mechanics in moving domains – Single continuum model

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Ph.D. Thesis



Monolithic Reformulation.

$$\mathbf{u} : \Omega \times [0, T] \rightarrow \mathbb{R}^3,$$

$$\mathbf{v} = \begin{cases} \mathbf{v}^s & \text{in } \Omega^s \\ \mathbf{v}^f & \text{in } \Omega^f \end{cases}$$

$$\mathbf{v} : \Omega \times [0, T] \rightarrow \mathbb{R}^3,$$

$$\mathbf{u} = \begin{cases} \mathbf{u}^s & \text{in } \Omega^s \\ \mathbf{u}^f & \text{in } \Omega^f. \end{cases}$$

$$\mathbf{F} = \mathbf{I} + \nabla_{\mathbf{x}} \mathbf{u}, \quad J = \det \mathbf{F}.$$

Lagrangian descr. In the solid part

$$\begin{aligned} J &= 1 && \text{in } \Omega^s, \\ J \rho^s \frac{\partial \mathbf{v}}{\partial t} &= \nabla_{\mathbf{x}} \cdot \mathbf{P}^s && \text{in } \Omega^s. \end{aligned}$$

ALE in the fluid part

$$\begin{aligned} 0 &= \nabla_{\mathbf{x}} \cdot (J \mathbf{v} \mathbf{F}^{-\top}), \\ 0 &= J \rho^f \frac{\partial \mathbf{v}}{\partial t} \Big|_{\mathbf{x}} + J \rho^f \nabla_{\mathbf{x}} \mathbf{v} \left(\mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}^f}{\partial t} \right) \right) - \nabla_{\mathbf{x}} \cdot (J \boldsymbol{\sigma} \mathbf{F}^{-\top}). \\ \frac{\partial \mathbf{u}}{\partial t} &= \Delta \mathbf{u} && \text{in } \Omega^f, \quad \mathbf{u} = \mathbf{u}^s \text{ on } \Gamma_t^r, \quad \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{fix}^s. \end{aligned}$$

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Ph.D. Thesis

Monolithic Reformulation.

$$\frac{\partial u}{\partial t} = \begin{cases} v & \text{in } \Omega^s \\ \Delta u & \text{in } \Omega^f \end{cases}$$

$$0 = \begin{cases} J - 1 & \text{in } \Omega^s \\ \nabla_{\mathbf{X}} \cdot (J \mathbf{v} \mathbf{F}^{-\top}) & \text{in } \Omega^f \end{cases}$$

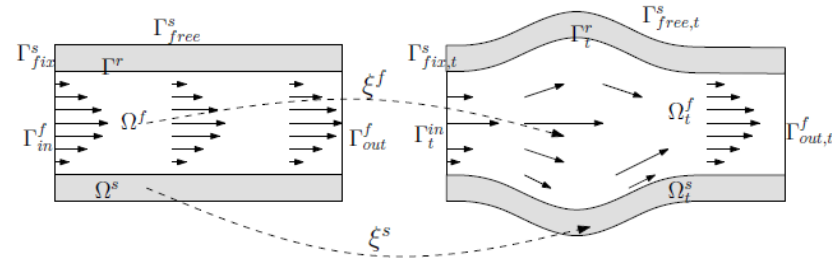
$$J \rho^s \frac{\partial \mathbf{v}}{\partial t} = \nabla_{\mathbf{X}} \cdot \mathbf{P}^s \quad \text{in } \Omega^s$$

$$J \rho^f \frac{\partial \mathbf{v}}{\partial t} = -J \rho^f (\nabla_{\mathbf{X}} \mathbf{v}) \mathbf{F}^{-1} \left(\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} \right) + \nabla_{\mathbf{X}} \cdot (J \boldsymbol{\sigma}^f \mathbf{F}^{-\top}) \quad \text{in } \Omega^f$$

initial conditions

$$\mathbf{u}(0) = \mathbf{0} \quad \text{in } \Omega,$$

$$\mathbf{v}(0) = \mathbf{v}_0 \quad \text{in } \Omega,$$



boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_{in}^f, \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{fix}^s, \boldsymbol{\sigma}^s \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{free}^s,$$

$$J \boldsymbol{\sigma} \mathbf{F}^{-\top} \mathbf{n}_{\mathbf{X}} = J \boldsymbol{\sigma} \mathbf{F}^{-\top} \mathbf{n}_{\mathbf{X}} \text{ on } \Gamma^r.$$